WEIGHTED RESIDUAL METHOD

INTRODUCTION

- Direct stiffness method is limited for simple 1D problems
- PMPE is limited to potential problems
- FEM can be applied to many engineering problems that are governed by a differential equation
- Need systematic approaches to generate FE equations
  - Weighted residual method
  - Energy method
- Ordinary differential equation (second-order or fourth-order) can be solved using the weighted residual method, in particular using Galerkin method

EXACT VS. APPROXIMATE SOLUTION

- Exact solution
  - Boundary value problem: differential equation + boundary conditions
  - Displacements in a uniaxial bar subject to a distributed force *p*(*x*)

$$\frac{d^2u}{dx^2} + p(x) = 0, \; 0 \le x \le 1$$

$$\left.\begin{aligned} u(0) &= 0 \\ \frac{du}{dx}(1) &= 1 \end{aligned}\right\} \text{ Boundary conditions}$$

  - Essential BC: The solution value at a point is prescribed (displacement or kinematic BC)
  - Natural BC: The derivative is given at a point (stress BC)
  - **Exact solution *u*(*x*)**: twice differential function
  - In general, it is difficult to find the exact solution when the domain and/or boundary conditions are complicated
  - Sometimes the solution may not exists even if the problem is well defined

EXACT VS. APPROXIMATE SOLUTION *cont.*

- Approximate solution
  - It satisfies the essential BC, but not natural BC
  - The approximate solution may not satisfy the DE exactly
  - ***Residual***: $\frac{d^2\tilde{u}}{dx^2} + p(x) = R(x)$
  - Want to minimize the ***residual*** by multiplying with a weight ***W*** and integrate over the domain

$$\int_0^1 R(x)W(x)dx = 0$$

Weight function

  - If it satisfies for any *W*(*x*), then *R*(*x*) will approaches zero, and the approximate solution will approach the exact solution
  - Depending on choice of *W*(*x*): least square error method, collocation method, Petrov-Galerkin method, and Galerkin method

GALERKIN METHOD

- Approximate solution is a linear combination of trial functions

$$\tilde{u}(x) = \sum_{i=1}^{N} c_i \phi_i(x)$$

Trial function

  - Accuracy depends on the choice of trial functions
  - The approximate solution must satisfy the essential BC
- Galerkin method
  - Use N trial functions for weight functions

$$\int_0^1 R(x)\phi_i(x)dx = 0, \quad i = 1,\ldots,N$$

$$\Longrightarrow \int_0^1 \left(\frac{d^2\tilde{u}}{dx^2} + p(x) \right) \phi_i(x)dx = 0, \quad i = 1,\ldots,N$$

$$\Longrightarrow \int_0^1 \frac{d^2\tilde{u}}{dx^2} \phi_i(x)dx = -\int_0^1 p(x)\phi_i(x)dx, \quad i = 1,\ldots,N$$

GALERKIN METHOD *cont*.

- Galerkin method cont.
  - Integration-by-parts: reduce the order of differentiation in u(x)

$$\frac{d\tilde{u}}{dx}\phi_i \Big|_0^1 - \int_0^1 \frac{d\tilde{u}}{dx}\frac{d\phi_i}{dx}dx = -\int_0^1 p(x)\phi_i(x)dx, \quad i = 1,\ldots,N$$

  - Apply natural BC and rearrange

$$\int_0^1 \frac{d\phi_i}{dx}\frac{d\tilde{u}}{dx}dx = \int_0^1 p(x)\phi_i(x)dx + \frac{du}{dx}(1)\phi_i(1) - \frac{du}{dx}(0)\phi_i(0), \quad i = 1,\ldots,N$$

  - Same order of differentiation for both trial function and approx. solution
  - Substitute the approximate solution

$$\int_0^1 \frac{d\phi_i}{dx} \sum_{j=1}^{N} c_j \frac{d\phi_j}{dx}dx = \int_0^1 p(x)\phi_i(x)dx + \frac{du}{dx}(1)\phi_i(1) - \frac{du}{dx}(0)\phi_i(0), \quad i = 1,\ldots,N$$

GALERKIN METHOD *cont*.

- Galerkin method cont.
  - Write in matrix form

$$\sum_{j=1}^{N} K_{ij} c_j = F_i, \quad i = 1, \ldots, N \quad \Longrightarrow \quad \underset{(N\times N)}{[\mathbf{K}]}\underset{(N\times 1)}{\{\mathbf{c}\}} = \underset{(N\times 1)}{\{\mathbf{F}\}}$$

$$K_{ij} = \int_0^1 \frac{d\phi_i}{dx}\frac{d\phi_j}{dx} dx$$

$$F_i = \int_0^1 p(x)\phi_i(x) dx + \frac{du}{dx}(1)\phi_i(1) - \frac{du}{dx}(0)\phi_i(0)$$

  - Coefficient matrix is symmetric; $K_{ij} = K_{ji}$
  - N equations with N unknown coefficients

EXAMPLE1

- Differential equation

$$\frac{d^2u}{dx^2} + 1 = 0, 0 \le x \le 1$$

$$\left.\begin{aligned} u(0) &= 0 \\ \frac{du}{dx}(1) &= 1 \end{aligned}\right\} \text{ Boundary conditions}$$

Trial functions

$$\phi_1(x) = x \qquad \phi_1'(x) = 1$$

$$\phi_2(x) = x^2 \qquad \phi_2'(x) = 2x$$

- Approximate solution (satisfies the essential BC)

$$\tilde{u}(x) = \sum_{i=1}^{2} c_i \phi_i(x) = c_1 x + c_2 x^2$$

- Coefficient matrix and RHS vector

$$K_{11} = \int_0^1 (\phi_1')^2 dx = 1$$

$$K_{12} = K_{21} = \int_0^1 (\phi_1'\phi_2') dx = 1$$

$$K_{22} = \int_0^1 (\phi_2')^2 dx = \frac{4}{3}$$

$$F_1 = \int_0^1 \phi_1(x) dx + \phi_1(1) - \cancel{\frac{du}{dx}}(0)\phi_1(0) = \frac{3}{2}$$

$$F_2 = \int_0^1 \phi_2(x) dx + \phi_2(1) - \cancel{\frac{du}{dx}}(0)\phi_2(0) = \frac{4}{3}$$

EXAMPLE1 *cont.*

- Matrix equation

$$[\mathbf{K}] = \frac{1}{3}\begin{bmatrix} 3 & 3 \\ 3 & 4 \end{bmatrix} \qquad \{\mathbf{F}\} = \frac{1}{6}\begin{Bmatrix} 9 \\ 8 \end{Bmatrix} \quad \Longrightarrow \quad \{\mathbf{c}\} = [\mathbf{K}]^{-1}\{\mathbf{F}\} = \begin{Bmatrix} 2 \\ -\frac{1}{2} \end{Bmatrix}$$

- Approximate solution

$$\tilde{u}(x) = 2x - \frac{x^2}{2}$$

  – Approximate solution is also the exact solution because the linear combination of the trial functions can represent the exact solution

EXAMPLE2

- Differential equation

$$\frac{d^2u}{dx^2} + x = 0, \qquad 0 \le x \le 1$$

$$\left.\begin{aligned} u(0) &= 0 \\ \frac{du}{dx}(1) &= 1 \end{aligned}\right\} \text{ Boundary conditions}$$

Trial functions

$$\phi_1(x) = x \qquad \phi_1'(x) = 1$$

$$\phi_2(x) = x^2 \qquad \phi_2'(x) = 2x$$

- Coefficient matrix is same, force vector: $\{\mathbf{F}\} = \frac{1}{12}\begin{Bmatrix} 16 \\ 15 \end{Bmatrix}$

$$\{\mathbf{c}\} = [\mathbf{K}]^{-1}\{\mathbf{F}\} = \begin{Bmatrix} \frac{19}{12} \\ -\frac{1}{4} \end{Bmatrix} \quad \Longrightarrow \quad \tilde{u}(x) = \frac{19}{12}x - \frac{x^2}{4}$$

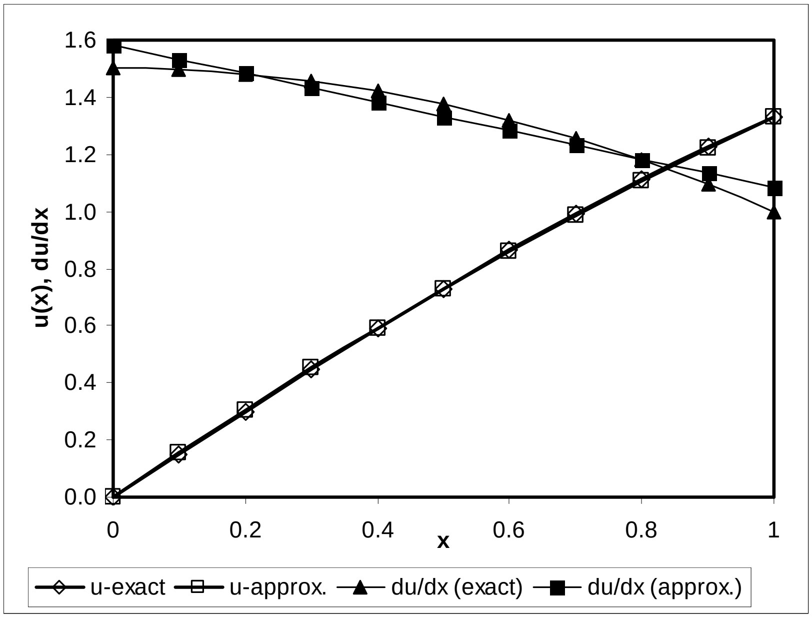
- Exact solution

$$u(x) = \frac{3}{2}x - \frac{x^3}{6}$$

  – The trial functions cannot express the exact solution; thus, approximate solution is different from the exact one

EXAMPLE2 *cont*.

- Approximation is good for $u(x)$, but not good for du/dx



HIGHER-ORDER DIFFERENTIAL EQUATIONS

- Fourth-order differential equation

$$\frac{d^4w}{dx^4} - p(x) = 0, \quad 0 \le x \le L$$

  - Beam bending under pressure load

$$\left.\begin{aligned} w(0) &= 0 \\ \frac{dw}{dx}(0) &= 0 \end{aligned}\right\} \text{Essential BC}$$

$$\left.\begin{aligned} \frac{d^2w}{dx^2}(L) &= M \\ \frac{d^3w}{dx^3}(L) &= -V \end{aligned}\right\} \text{Natural BC}$$

- Approximate solution

$$\tilde{w}(x) = \sum_{i=1}^{N} c_i \phi_i(x)$$

- Weighted residual equation (Galerkin method)

$$\int_0^L \left(\frac{d^4\tilde{w}}{dx^4} - p(x) \right) \phi_i(x) dx = 0, \quad i = 1, \ldots, N$$

  - In order to make the order of differentiation same, integration-by-parts must be done twice

HIGHER-ORDER DE *cont.*

- After integration-by-parts twice

$$\frac{d^3\tilde{w}}{dx^3}\phi_i\bigg|_0^L - \frac{d^2\tilde{w}}{dx^2}\frac{d\phi_i}{dx}\bigg|_0^L + \int_0^L \frac{d^2\tilde{w}}{dx^2}\frac{d^2\phi_i}{dx^2}dx = \int_0^L p(x)\phi_i(x)dx, \quad i = 1,\dots,N$$

$$\Rightarrow \int_0^L \frac{d^2\tilde{w}}{dx^2}\frac{d^2\phi_i}{dx^2}dx = \int_0^L p(x)\phi_i(x)dx - \frac{d^3\tilde{w}}{dx^3}\phi_i\bigg|_0^L + \frac{d^2\tilde{w}}{dx^2}\frac{d\phi_i}{dx}\bigg|_0^L, \quad i = 1,\dots,N$$

- Substitute approximate solution

$$\int_0^L \sum_{j=1}^N c_j \frac{d^2\phi_j}{dx^2}\frac{d^2\phi_i}{dx^2}dx = \int_0^L p(x)\phi_i(x)dx - \frac{d^3\tilde{w}}{dx^3}\phi_i\bigg|_0^L + \frac{d^2\tilde{w}}{dx^2}\frac{d\phi_i}{dx}\bigg|_0^L, \quad i = 1,\dots,N$$

  - Do not substitute the approx. solution in the boundary terms

- Matrix form

$$\underset{N\times N}{[\mathbf{K}]}\underset{N\times 1}{\{\mathbf{c}\}} = \underset{N\times 1}{\{\mathbf{F}\}}$$

$$K_{ij} = \int_0^L \frac{d^2\phi_i}{dx^2}\frac{d^2\phi_j}{dx^2}dx$$

$$F_i = \int_0^L p(x)\phi_i(x)dx - \frac{d^3w}{dx^3}\phi_i\bigg|_0^L + \frac{d^2w}{dx^2}\frac{d\phi_i}{dx}\bigg|_0^L$$

EXMAPLE

- Fourth-order DE

$$\frac{d^4w}{dx^4} - 1 = 0, \quad 0 \le x \le L$$

$$w(0) = 0 \quad \frac{dw}{dx}(0) = 0$$

$$\frac{d^2w}{dx^2}(1) = 2 \quad \frac{d^3w}{dx^3}(1) = -1$$

- Two trial functions

$$\phi_1 = x^2, \quad \phi_2 = x^3 \qquad \phi_1'' = 2, \quad \phi_2'' = 6x$$

- Coefficient matrix

$$K_{11} = \int_0^1 (\phi_1'')^2 dx = 4$$

$$K_{12} = K_{21} = \int_0^1 (\phi_1''\phi_2'')dx = 6 \quad \Rightarrow \quad [\mathbf{K}] = \begin{bmatrix} 4 & 6 \\ 6 & 12 \end{bmatrix}$$

$$K_{22} = \int_0^1 (\phi_2'')^2 dx = 12$$

EXAMPLE *cont*.

- RHS

$$F_1 = \int_0^1 x^2 dx + V\phi_1(1) + \cancel{\frac{d^3w(0)}{dx^3}\phi_1(0)} + M\phi_1'(1) - \cancel{\frac{d^2w(0)}{dx^2}\phi_1'(0)} = \frac{16}{3}$$

$$F_2 = \int_0^1 x^3 dx + V\phi_2(1) + \cancel{\frac{d^3w(0)}{dx^3}\phi_2(0)} + M\phi_2'(1) - \cancel{\frac{d^2w(0)}{dx^2}\phi_2'(0)} = \frac{29}{4}$$

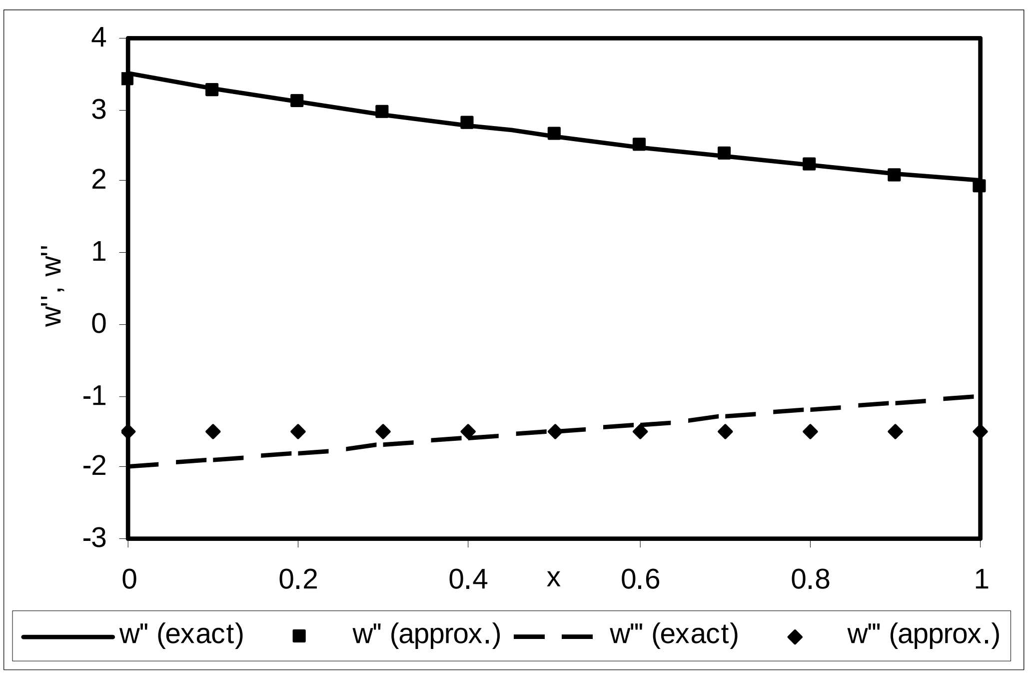
- Approximate solution

$$\{\mathbf{c}\} = [\mathbf{K}]^{-1}\{\mathbf{F}\} = \begin{Bmatrix} \frac{41}{24} \\ -\frac{1}{4} \end{Bmatrix} \Longrightarrow \tilde{w}(x) = \frac{41}{24}x^2 - \frac{1}{4}x^3$$

- Exact solution

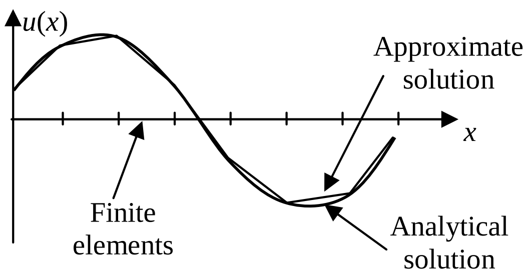
$$w(x) = \frac{1}{24}x^4 - \frac{1}{3}x^3 + \frac{7}{4}x^2$$

EXAMPLE *cont*.



FINITE ELEMENT APPROXIMATION

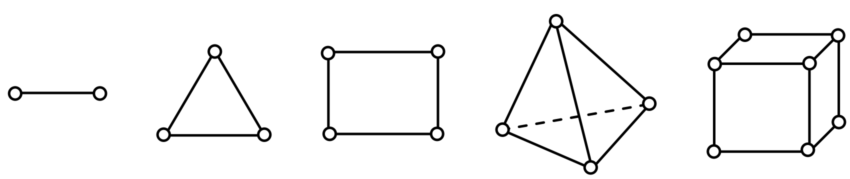
- Domain Discretization
  - Weighted residual method is still difficult to obtain the trial functions that satisfy the essential BC
  - FEM is to divide the entire domain into a set of simple sub-domains (finite element) and share nodes with adjacent elements
  - Within a finite element, the solution is approximated in a simple polynomial form



  - When more number of finite elements are used, the approximated piecewise linear solution may converge to the analytical solution

FINITE ELEMENT METHOD *cont*.

- Types of finite elements



1D 2D 3D

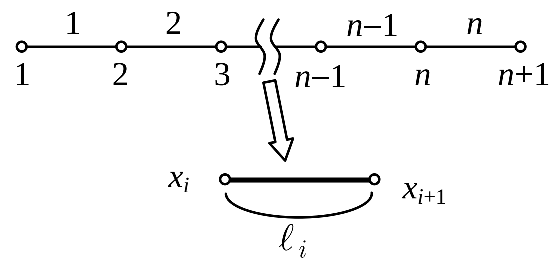
- Variational equation is imposed on each element.

$$\int_0^1 \Box\, dx = \int_0^{0.1} \Box\, dx + \int_{0.1}^{0.2} \Box\, dx + \cdots + \int_{0.9}^{1} \Box\, dx$$

One element

TRIAL SOLUTION

– Solution within an element is approximated using simple polynomials.



– *i*-th element is composed of two nodes: x_i and x_{i+1}. Since two unknowns are involved, linear polynomial can be used:

$$\tilde{u}(x) = a_0 + a_1 x, \quad x_i \le x \le x_{i+1}$$

– The unknown coefficients, a_0 and a_1, will be expressed in terms of nodal solutions $u(x_i)$ and $u(x_{i+1})$.

TRIAL SOLUTION *cont*.

– Substitute two nodal values

$$\begin{cases} \tilde{u}(x_i) = u_i = a_0 + a_1 x_i \\ \tilde{u}(x_{i+1}) = u_{i+1} = a_0 + a_1 x_{i+1} \end{cases}$$

– Express a_0 and a_1 in terms of u_i and u_{i+1}. Then, the solution is approximated by

$$\tilde{u}(x) = \underbrace{\frac{x_{i+1} - x}{L^{(i)}}}_{N_i(x)} u_i + \underbrace{\frac{x - x_i}{L^{(i)}}}_{N_{i+1}(x)} u_{i+1}$$

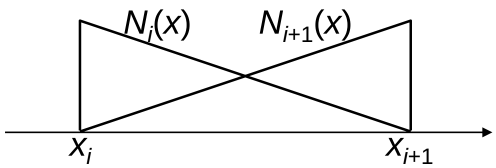
– Solution for i-th element:

$$\tilde{u}(x) = N_i(x) u_i + N_{i+1}(x) u_{i+1}, \quad x_i \le x \le x_{i+1}$$

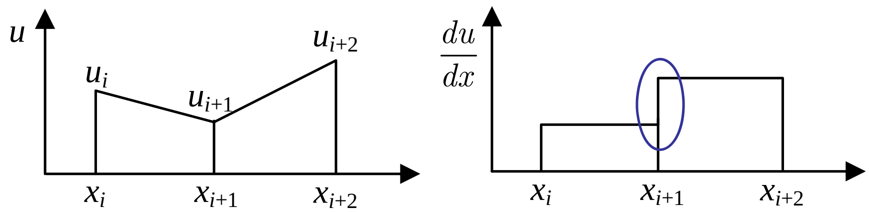
– $N_i(x)$ and $N_{i+1}(x)$: **Shape Function** or **Interpolation Function**

TRIAL SOLUTION *cont*.

- Observations
  - Solution $u(x)$ is interpolated using its nodal values u_i and u_{i+1}.
  - $N_i(x) = 1$ at node x_i, and $=0$ at node x_{i+1}.



  - The solution is approximated by piecewise linear polynomial and its gradient is constant within an element.



  - Stress and strain (derivative) are often averaged at the node.

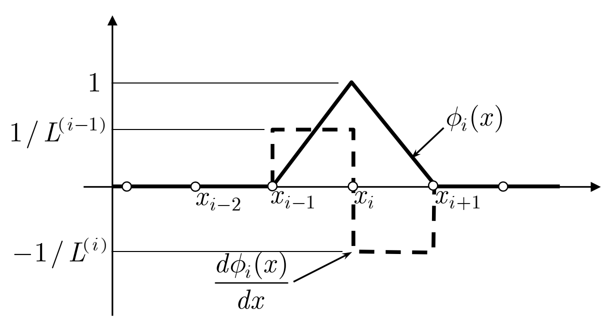
GALERKIN METHOD

- Relation between interpolation functions and trial functions
  - 1D problem with linear interpolation

$$\tilde{u}(x) = \sum_{i=1}^{N_D} u_i \phi_i(x) \qquad \phi_i(x) = \begin{cases} 0, & 0 \le x \le x_{i-1} \\ N_i^{(i-1)}(x) = \dfrac{x - x_{i-1}}{L^{(i-1)}}, & x_{i-1} < x \le x_i \\ N_i^{(i)}(x) = \dfrac{x_{i+1} - x}{L^{(i)}}, & x_i < x \le x_{i+1} \\ 0, & x_{i+1} < x \le x_{N_D} \end{cases}$$

  - Difference: the interpolation function does not exist in the entire domain, but it exists only in elements connected to the node
- Derivative

$$\frac{d\phi_i(x)}{dx} = \begin{cases} 0, & 0 \le x \le x_{i-1} \\ \dfrac{1}{L^{(i-1)}}, & x_{i-1} < x \le x_i \\ -\dfrac{1}{L^{(i)}}, & x_i < x \le x_{i+1} \\ 0, & x_{i+1} < x \le x_{N_D} \end{cases}$$



EXAMPLE

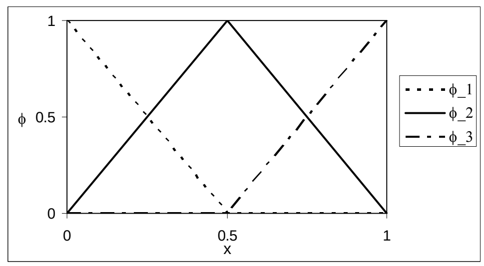
- Solve using two equal-length elements

$$\frac{d^2u}{dx^2} + 1 = 0, 0 \le x \le 1 \qquad \left.\begin{matrix} u(0) = 0 \\ \frac{du}{dx}(1) = 1 \end{matrix}\right\} \text{ Boundary conditions}$$

- Three nodes at x = 0, 0.5, 1.0; displ at nodes = u_1, u_2, u_3
- Approximate solution $\tilde{u}(x) = u_1\phi_1(x) + u_2\phi_2(x) + u_3\phi_3(x)$

$$\phi_1(x) = \begin{cases} 1 - 2x, & 0 \le x \le 0.5 \\ 0, & 0.5 < x \le 1 \end{cases} \qquad \phi_2(x) = \begin{cases} 2x, & 0 \le x \le 0.5 \\ 2 - 2x, & 0.5 < x \le 1 \end{cases}$$

$$\phi_3(x) = \begin{cases} 0, & 0 \le x \le 0.5 \\ -1 + 2x, & 0.5 < x \le 1 \end{cases}$$



EXAMPLE *cont*.

- Derivatives of interpolation functions

$$\frac{d\phi_1(x)}{dx} = \begin{cases} -2, & 0 \le x \le 0.5 \\ 0, & 0.5 < x \le 1 \end{cases} \qquad \frac{d\phi_2(x)}{dx} = \begin{cases} 2, & 0 \le x \le 0.5 \\ -2, & 0.5 < x \le 1 \end{cases}$$

$$\frac{d\phi_3(x)}{dx} = \begin{cases} 0, & 0 \le x \le 0.5 \\ 2, & 0.5 < x \le 1 \end{cases}$$

- Coefficient matrix

$$K_{12} = \int_0^1 \frac{d\phi_1}{dx}\frac{d\phi_2}{dx} dx = \int_0^{0.5} (-2)(2) dx + \int_{0.5}^1 (0)(-2) dx = -2$$

$$K_{22} = \int_0^1 \frac{d\phi_2}{dx}\frac{d\phi_2}{dx} dx = \int_0^{0.5} 4 dx + \int_{0.5}^1 4 dx = 4$$

- RHS

$$F_1 = \int_0^{0.5} 1 \times (1 - 2x) dx + \int_{0.5}^1 1 \times (0) dx + \frac{du}{dx}(1)\phi_1(1) - \frac{du}{dx}(0)\phi_1(0) = 0.25 - \frac{du}{dx}(0)$$

$$F_2 = \int_0^{0.5} 2x dx + \int_{0.5}^1 (2 - 2x) dx + \frac{du}{dx}(1)\phi_2(1) - \frac{du}{dx}(0)\phi_2(0) = 0.5$$

EXAMPLE *cont.*

- Matrix equation

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0.5 \\ 1.25 \end{Bmatrix}$$

← Consider it as unknown (F_1)

- Striking the 1st row and striking the 1st column (BC)

$$\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0.5 \\ 1.25 \end{Bmatrix}$$

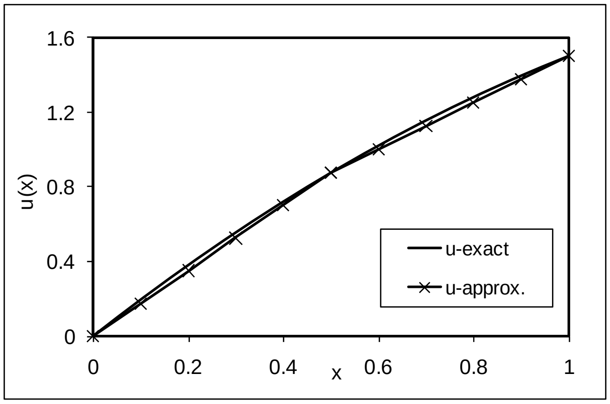
- Solve for $u_2 = 0.875$, $u_3 = 1.5$
- Approximate solution

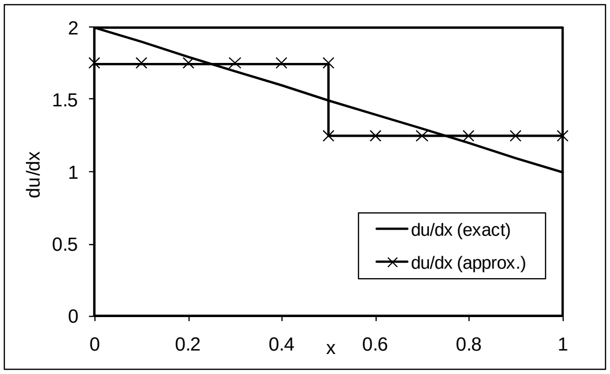
$$\tilde{u}(x) = \begin{cases} 1.75x, & 0 \le x \le 0.5 \\ 0.25 + 1.25x, & 0.5 \le x \le 1 \end{cases}$$

  - Piecewise linear solution

EXAMPLE *cont.*

- Solution comparison
- Approx. solution has about 8% error
- Derivative shows a large discrepancy
- Approx. derivative is constant as the solution is piecewise linear





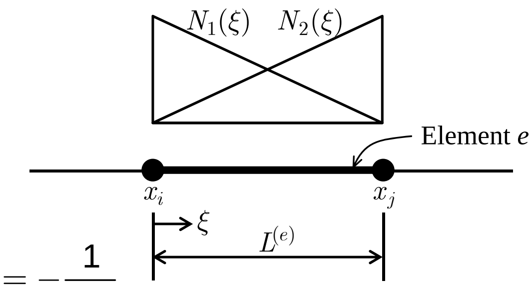
FORMAL PROCEDURE

- Galerkin method is still not general enough for computer code
- Apply Galerkin method to one element (e) at a time
- Introduce a local coordinate

$$x = x_i(1-\xi) + x_j\xi \qquad \xi = \frac{x - x_i}{x_j - x_i} = \frac{x - x_i}{L^{(e)}}$$

- Approximate solution within the element

$$\tilde{u}(x) = u_i N_1(x) + u_j N_2(x)$$



$$N_1(\xi) = (1-\xi)$$
$$N_2(\xi) = \xi$$

$$N_1(x) = \left(1 - \frac{x - x_i}{L^{(e)}}\right) \qquad \frac{dN_1}{dx} = \frac{dN_1}{d\xi}\frac{d\xi}{dx} = -\frac{1}{L^{(e)}}$$

$$N_2(x) = \frac{x - x_i}{L^{(e)}} \qquad \frac{dN_2}{dx} = \frac{dN_2}{d\xi}\frac{d\xi}{dx} = +\frac{1}{L^{(e)}}$$

FORMAL PROCEDURE *cont.*

- Interpolation property

$$N_1(x_i) = 1, \quad N_1(x_j) = 0 \qquad \tilde{u}(x_i) = u_i$$
$$N_2(x_i) = 0, \quad N_2(x_j) = 1 \qquad \tilde{u}(x_j) = u_j$$

- Derivative of approx. solution

$$\frac{d\tilde{u}}{dx} = u_i\frac{dN_1}{dx} + u_j\frac{dN_2}{dx}$$

$$\frac{d\tilde{u}}{dx} = \left[\frac{dN_1}{dx} \quad \frac{dN_2}{dx}\right]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{L^{(e)}}\left[\frac{dN_1}{d\xi} \quad \frac{dN_2}{d\xi}\right]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

- Apply Galerkin method in the element level

$$\int_{x_i}^{x_j} \frac{dN_i}{dx}\frac{d\tilde{u}}{dx}dx = \int_{x_i}^{x_j} p(x)N_i(x)dx + \frac{du}{dx}(x_j)N_i(x_j) - \frac{du}{dx}(x_i)N_i(x_i), \quad i = 1,2$$

FORMAL PROCEDURE *cont.*

- Change variable from x to ξ

$$\frac{1}{L^{(e)}}\int_0^1 \frac{dN_i}{d\xi}\left[\frac{dN_1}{d\xi} \quad \frac{dN_2}{d\xi}\right]d\xi \bullet \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = L^{(e)}\int_0^1 p(x)N_i(\xi)d\xi$$

$$+\frac{du}{dx}(x_j)N_i(1) - \frac{du}{dx}(x_i)N_i(0), \quad i = 1,2$$

  - Do not use approximate solution for boundary terms
- Element-level matrix equation

$$[\mathbf{k}^{(e)}]\{\mathbf{u}^{(e)}\} = \left\{\mathbf{f}^{(e)}\right\} + \begin{Bmatrix} -\dfrac{du}{dx}(x_i) \\ +\dfrac{du}{dx}(x_j) \end{Bmatrix} \qquad \{\mathbf{f}^{(e)}\} = L^{(e)}\int_0^1 p(x)\begin{Bmatrix} N_1(\xi) \\ N_2(\xi) \end{Bmatrix}d\xi$$

$$\left[\underset{2\times 2}{\mathbf{k}^{(e)}}\right] = \frac{1}{L^{(e)}}\int_0^1 \begin{bmatrix} \left(\dfrac{dN_1}{d\xi}\right)^2 & \dfrac{dN_1}{d\xi}\dfrac{dN_2}{d\xi} \\ \dfrac{dN_2}{d\xi}\dfrac{dN_1}{d\xi} & \left(\dfrac{dN_2}{d\xi}\right)^2 \end{bmatrix} d\xi = \frac{1}{L^{(e)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

FORMAL PROCEDURE *cont.*

- Need to derive the element-level equation for all elements
- Consider Elements 1 and 2 (connected at Node 2)

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}^{(1)} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}^{(1)} + \begin{Bmatrix} -\dfrac{du}{dx}(x_1) \\ +\dfrac{du}{dx}(x_2) \end{Bmatrix}$$

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}^{(2)} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_2 \\ f_3 \end{Bmatrix}^{(2)} + \begin{Bmatrix} -\dfrac{du}{dx}(x_2) \\ +\dfrac{du}{dx}(x_3) \end{Bmatrix}$$

- Assembly

$$\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} \\ 0 & k_{21}^{(2)} & k_{22}^{(2)} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix} + \begin{Bmatrix} -\dfrac{du}{dx}(x_1) \\ 0 \\ \dfrac{du}{dx}(x_3) \end{Bmatrix}$$

Vanished unknown term (→ 0)

FORMAL PROCEDURE *cont.*

- Assembly of N_E elements ($N_D = N_E + 1$)

$$\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & \cdots & 0 \\ k_{21}^{(1)} & k_{22}^{(1)}+k_{11}^{(2)} & k_{12}^{(2)} & \cdots & 0 \\ 0 & k_{221}^{(2)} & k_{22}^{(2)}+k_{11}^{(2)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & k_{21}^{(N_E)} & k_{22}^{(N_E)} \end{bmatrix}_{(N_D\times N_D)} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_N \end{Bmatrix}_{(N_D\times 1)} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)}+f_2^{(2)} \\ f_3^{(2)}+f_3^{(3)} \\ \vdots \\ f_N^{(N_E)} \end{Bmatrix}_{(N_D\times 1)} + \begin{Bmatrix} -\frac{du}{dx}(x_1) \\ 0 \\ 0 \\ \vdots \\ +\frac{du}{dx}(x_N) \end{Bmatrix}_{(N_D\times 1)}$$

$$[\mathbf{K}]\{\mathbf{q}\} = \{\mathbf{F}\}$$

- Coefficient matrix [K] is singular; it will become non-singular after applying boundary conditions

EXAMPLE

- Use three equal-length elements

$$\frac{d^2u}{dx^2} + x = 0, \quad 0 \le x \le 1 \qquad u(0) = 0,\ u(1) = 0$$

- All elements have the same coefficient matrix

$$\left[\mathbf{k}^{(e)}\right]_{2\times 2} = \frac{1}{L^{(e)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}, \quad (e = 1,2,3)$$

- Change variable of $p(x) = x$ to $p(\xi)$: $\quad p(\xi) = x_i(1-\xi) + x_j\xi$
- RHS

$$\{\mathbf{f}^{(e)}\} = L^{(e)}\int_0^1 p(x)\begin{Bmatrix} N_1(\xi) \\ N_2(\xi) \end{Bmatrix} d\xi = L^{(e)}\int_0^1 [x_i(1-\xi) + x_j\xi]\begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix} d\xi$$

$$= L^{(e)}\begin{Bmatrix} \frac{x_i}{3} + \frac{x_j}{6} \\ \frac{x_i}{6} + \frac{x_j}{3} \end{Bmatrix}, \qquad (e = 1,2,3)$$

EXAMPLE *cont.*

- RHS cont. $\begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{Bmatrix} = \dfrac{1}{54}\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}, \quad \begin{Bmatrix} f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix} = \dfrac{1}{54}\begin{Bmatrix} 4 \\ 5 \end{Bmatrix}, \quad \begin{Bmatrix} f_3^{(3)} \\ f_4^{(3)} \end{Bmatrix} = \dfrac{1}{54}\begin{Bmatrix} 7 \\ 8 \end{Bmatrix}$
- Assembly

$$\begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 3+3 & -3 & 0 \\ 0 & -3 & 3+3 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{54} - \frac{du}{dx}(0) \\ \frac{2}{54} + \frac{4}{54} \\ \frac{7}{54} + \frac{5}{54} \\ \frac{8}{54} + \frac{du}{dx}(1) \end{Bmatrix}$$

Element 1
Element 2
Element 3

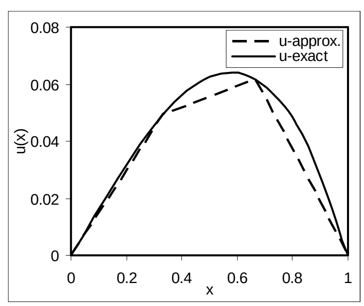
- Apply boundary conditions
  - Deleting 1st and 4th rows and columns

$$\begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \frac{1}{9}\begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \Longrightarrow \begin{matrix} u_2 = \frac{4}{81} \\ u_3 = \frac{5}{81} \end{matrix}$$

EXAMPLE *cont.*

- Approximate solution

$$\tilde{u}(x) = \begin{cases} \dfrac{4}{27}x, & 0 \le x \le \dfrac{1}{3} \\ \dfrac{4}{81} + \dfrac{1}{27}\left(x - \dfrac{1}{3}\right), & \dfrac{1}{3} \le x \le \dfrac{2}{3} \\ \dfrac{5}{81} - \dfrac{5}{27}\left(x - \dfrac{2}{3}\right), & \dfrac{2}{3} \le x \le 1 \end{cases}$$



- Exact solution

$$u(x) = \frac{1}{6}x\left(1 - x^2\right)$$

  - Three element solutions are poor
  - Need more elements

CONVERGENCE

- Weighted residual of 2*m*-th order DE has highest derivatives of order *m*
- With exact arithmetic, the following is sufficient for convergence to true solution (ϕ) as mesh is refined:
  - Complete polynomials of at least order *m* inside element
  - Continuity across element boundaries up to derivatives of order *m-1*
  - Element must be capable of representing exactly uniform ϕ and uniform derivatives up to order *m-1.*
- Beam: 4-th order DE (m = 2)
  - Complete polynomials: $v(x) = a_0 + a_1x + a_2x^2 + a_3x^3$
  - Continuity on $v(x)$ and $dv(x)/dx$ across element boundaries
  - Uniform $v(x) = a_0$
  - Uniform derivative $dv(x)/dx = a_1$

Beam elements will converge upon refinement

RIGID BODY MOTION

- Rigid body motion for CST can lead to non zero strains!

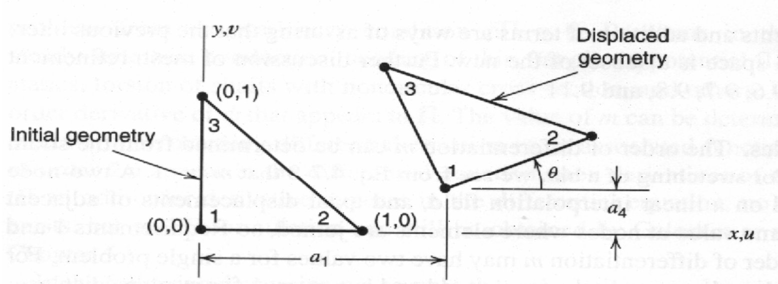


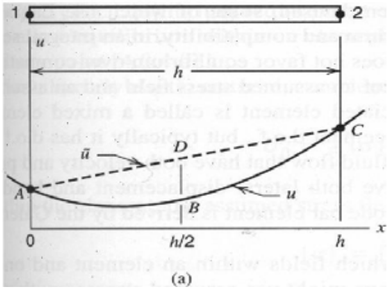
Figure 4.9-1. Constant strain triangle element subjected to a rigid body motion consisting of translations a_1 and a_4 in the x and y directions, and rotation θ about node 1.

- Rigid body motion

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} a_1 & \cos\theta - 1 & -\sin\theta \\ a_4 & \sin\theta & \cos\theta - 1 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

- The normal strain $\varepsilon_x = \dfrac{\partial u}{\partial x} = \cos\theta - 1 \neq 0$

CONVERGENCE RATE



Quadratic curve $u=a+bx+cx^2$ modeled by linear FE $u_{fe}=a+(b+ch)x$

- Maximal error at mid-point D

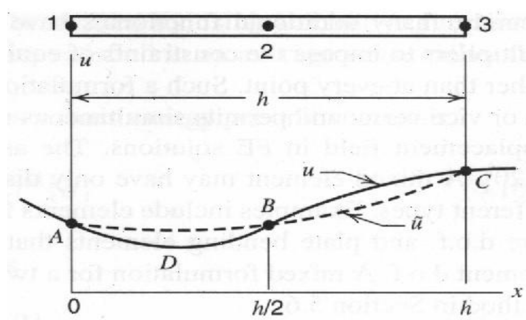
$$e_D = u_D - u_B = \frac{u_A + u_C}{2} - u_B = \frac{ch^2}{4} = \frac{h^2}{8} u''$$

- Maximal gradient error is maximal at ends

$$e'_A = \frac{u_C - u_A}{h} - b = hc = \frac{h}{2} u''$$

- Error in function converges faster than in derivative!

QUADRATIC ELEMENT FOR CUBIC SOLUTION



- Exact solution

$$u = a + bx + cx^2 + dx^3$$

- Finite element approximation

$$\tilde{u} = a + \left(b - \frac{1}{2} dh^2\right) x + \left(c + \frac{3}{2} dh\right) x^2$$

- Maximal errors

$$e_D = -\frac{3dh^3}{64} = -\frac{h^3}{128} u''' \qquad and \qquad e'_A = -\frac{dh^2}{2} = -\frac{h^2}{12} u'''$$

CONVERGENCE RATE

- Useful to know convergence rate
  - Estimate how much to refine
  - Detect modeling crimes
  - Extrapolate
- Most studies just do series of refinements if anything