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Match-Point Solutions for Robust Flutter Analysis

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The computation of robust flutter speeds presents a significant advancement over traditional types of flutter analysis. In particular, μ-method analysis is able to generate robust flutter speeds that represent worst-case flight conditions with respect to potential modeling errors. Robust flutter speeds may be computed using a model formulation that has been previously presented; however, that formulation has limitations in its ability to generate a match-point solution. A model formulation is introduced for which μ-method analysis is guaranteed to compute a match-point solution. The match-point solution is immediately realized by analyzing a single model so the computation time is reduced from the previous approach that required iterations. Also, the solution is able to consider parametric uncertainty in any element, whereas the previous formulation did not consider mass uncertainty. The match-point formulation is derived by properly treating the nonlinear perturbations and uncertainties that affect the equation of motion. The Aerostructures Test Wing is used to demonstrate that the μ-method analysis computes match-point flutter speeds using this new formulation.

Nomenclature

A = aerodynamic force matrix
$\boldsymbol{a}$ = scaled force vector
C = damping matrix
K = stiffness matrix
L = aerodynamic force model
M = mass matrix
n = number of modes
P = plant model
p = coefficient in density approximation
Q = aerodynamic force matrix
$\bar{Q}$ = force model
$\boldsymbol{q}$ = unscaled force vector
$\bar{q}$ = dynamic pressure
S = structural dynamic model
V = velocity
W = weighting matrix
w = input from uncertainty
z = output to uncertainty
β = lag pole
Δ = uncertainty matrix
δ = uncertainty parameter

Introduction

THE study of aeroelasticity has been an important discipline since the advent of flight. In particular, one of the primary goals of aeroelastic analysis is to predict the flight conditions associated with the onset of an instability called flutter.[1] The mechanism associated with this instability may be somewhat complex; however, there are often some basic characteristics associated with it. A common type of flutter is characterized by a phased coupling of modes in which damping for one mode decreases as speed increases near the onset of the instability. The extreme case, called explosive flutter, occurs when this decrease in damping occurs suddenly and dramatically.

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The flight conditions associated with flutter, known as flutter speeds, are difficult to predict. It is inherently dangerous to attempt to determine these conditions experimentally during flight testing because of the potentially unforeseen decrease in damping that may suddenly occur. Such behaviors limit the efficiency of flight testing so that it is vital that advanced analytical methods be developed to predict the onset of flutter.

The μ method is an advanced technique to analyze aeroelastic dynamics and to predict the onset of flutter.[2] This method is fundamentally different from traditional approaches. One difference is that it is based on concepts from robust control theory.[3] A second difference is that it is able to utilize both theoretical models and flight data. A third, and most important, difference is that μ-method analysis introduces the concept of a robust flutter speed.

The basic concept of μ-method analysis is to compute the smallest perturbation to a flight condition that incurs flutter. The robust nature of the solution results by considering the effect of modeling error during this computation. A traditional flutter speed can be interpreted as an indication of the flight conditions at which a model is no longer stable. A robust flutter speed can be interpreted as an indication of the flight conditions at which a model is no longer robustly stable.

A model formulation that considers perturbations to dynamic pressure was originally derived for μ-method analysis[2]; however, this formulation has limitations. The most important limitation in the original formulation is that the μ-method analysis of these models is not guaranteed to compute match-point flutter speeds. The robust analysis computes a worst-case perturbation to a single parameter; however, the model is actually a function of both dynamic pressure and airspeed. This implies that the model needs to consider multiple perturbations, or uncertainties, that are nonlinearly related. The original formulation only considers a perturbation to dynamic pressure and ignores the required corresponding perturbation to airspeed. Thus, the resulting solution is a dynamic pressure for a model with a nonmatch-point airspeed.

This paper introduces a model formulation that results in match-point solutions from μ-method analysis. The main difference between this formulation and the original formulation is the inclusion of nonlinear uncertainties that affect coupled parameters in the equation of motion. In particular, the equation of motion is shown to be a polynomial in terms of the uncertainties and, thus, can be written in the μ-method framework.[4] The model formulation uses feedback loops that are coupled through the feedthrough matrix of the plant to account for the nonlinearities in the uncertain parameters.[5,6]

The key to getting a model that results in match-point flutter speeds is to formulate this model so that a single parameter describes the flight condition. This allows the μ-method analysis to find the worst-case perturbation to that parameter without considering if there are related parameters that must be analyzed. The formulation replaces the dynamic pressure with a polynomial function of airspeed. Thus, the flight condition of the model is expressed entirely through a dependency on airspeed.

Note that this paper does not introduce changes to the basic concept of μ-method analysis; rather, it merely introduces a new modeling formulation to be used with that analysis. The same definitions and algorithms that compute robust flutter speeds are used for both the original and the new formulations. The limitations in the original models, not the μ method, are eliminated by the results in this paper.

Furthermore, the μ method has several applications besides the analysis of aeroelastic models. An online approach to use μ-method analysis during flight testing is the basis for a tool called the flutterometer.[7] Another application is the analysis of aeroservoelasticity for flight vehicles.[8] This paper considers the analysis of flutter; however, the match-point formulation may be directly used for all of the applications of μ-method analysis.

Aeroelastic Dynamics

Equation of Motion

The nominal dynamics of an aeroelastic system at constant Mach number are described by the standard equation of motion as given in matrix–vector notation

$$M\ddot{\eta} + C\dot{\eta} + K\eta + \bar{q}Q\eta = 0 \tag{1}$$

The equation of motion uses $\eta \in \boldsymbol{R}^n$ to represent the elastic displacements of a system with n modes. The structural dynamics of the system are represented by $M \in \boldsymbol{R}^{n\times n}$ as the mass, $C \in \boldsymbol{R}^{n\times n}$ as the damping, and $K \in \boldsymbol{R}^{n\times n}$ as the stiffness. The aerodynamics of the system are represented in the equation of motion as $\bar{q} \in \boldsymbol{R}$ for the dynamic pressure and $Q \in \boldsymbol{C}^{n\times n}$ for the unsteady aerodynamic forces.

The aerodynamic forces Q are typically determined as a set of frequency-dependent matrices. A set of these matrices are computed to represent the forces at a number of distinct values of reduced frequency. As such, the forces are not predicted as a closed-form analytical solution and are not suitable for state-space representation and μ-method analysis.

There are several representations that may be used to describe the unsteady aerodynamic forces. These formulations, often referred to as rational function approximations, are essentially transfer functions that may be expressed analytically. This paper uses a particular formulation, often referred to as Roger's formulation,[9] to represent the forces as

$$Q = A_0 + A_1 ik + A_2(ik)^2 + A_3[ik/(ik+\beta_1)] + A_4[ik/(ik+\beta_2)] \tag{2}$$

There are essentially two types of forces in Eq. (2). One type of forces result from the quasi-steady aerodynamics. They are represented by $A_0 \in \boldsymbol{R}^{n\times n}$ as the steady forces that act like an equivalent aerodynamic stiffness, $A_1 \in \boldsymbol{R}^{n\times n}$ as the forces that act like an aerodynamic damping, and $A_2 \in \boldsymbol{R}^{n\times n}$ as the forces that act like an aerodynamic inertia. The other type of forces consists of the purely unsteady aerodynamic lags that are represented by Padé approximates (see Ref. 10). There are scaling matrices, A_3, $A_4 \in \boldsymbol{R}^{n\times n}$, and poles, β_1, $\beta_2 \in \boldsymbol{R}$, used to define the effects of each lag. The scaling matrices are often computed such that the quasi-steady and purely unsteady effects may not be completely separated[11]; however, the results in this paper are not dependent on such separation and so Eq. (2) is assumed to be valid without loss of generality.

Equation (2) is formulated with two lag terms, but this is not a requirement. In practice, some sets of force matrices may be better represented by using a single lag term, whereas other sets may require many lag terms. The model formulation in this paper uses two lag terms, but it is noted in the appropriate areas how the derivations are extended to include fewer or more lag terms.

Also, there are many methods to compute the coefficient matrices in Eq. (2). One such method uses a straightforward least-squares approach that can incorporate flight data.[12] Another common method uses a minimum-state approximation that may reduce model complexity.[13] Other methods, including pure-lag models[14] or combinations of pure-lag models with minimum-state approximations,[15] can be used. The μ-method analysis is not restricted to any particular method for choosing the model elements; instead, the computation of robust flutter margins is actually more sensitive to the number of uncertainties rather than the number of states in the state-space approximation.

The frequency-varying nature of the unsteady aerodynamics is provided in Roger's[9] form by the dependency on $k \in \boldsymbol{R}$. This variable is the reduced frequency of the system and can be related to the traditional Laplace variable, $s \in \boldsymbol{C}$. The relationship uses a reference length, $b \in \boldsymbol{R}$, and an airspeed, $V \in \boldsymbol{R}$, as scaling parameters such that $ik = (b/V)s$.

The equation of motion for a general aeroelastic system can be written by combining Eqs. (1) and (2) with the definition for k. The resulting system is described as follows and is used throughout this paper as the baseline model:

$$M\ddot{\eta} + C\dot{\eta} + K\eta = -\bar{q}\left[A_0 + A_1\frac{b}{V}s + A_2\frac{b^2}{V^2}s^2 + A_3\frac{(b/V)s}{(b/V)s+\beta_1} + A_4\frac{(b/V)s}{(b/V)s+\beta_2}\right]\eta \tag{3}$$

Parametric Uncertainty

There are several types of uncertainty operators that may be used by the μ method to describe modeling errors[2]; however, this paper will restrict the analysis to consider only parametric uncertainty. Parametric uncertainty refers to operators that directly account for errors in particular parameters in the equation of motion. This type of uncertainty is typically represented by block-diagonal matrices that are real and constant.

Consider a parametric uncertainty, $\Delta_M \in \boldsymbol{R}^{n\times n}$, that is associated with the mass matrix. The mass in the equation of motion, M, is actually assumed to be some nominal value, M_0, plus the weighted uncertainty operator. The μ-method considers only uncertainty operators that have an $\mathcal{H}_\infty$ norm less than unity; thus, the weightings are needed to scale the unity-bound operator and admit the desired magnitude of modeling error. The uncertain mass is

$$M = M_0 + M_1 W_M \Delta_M \tag{4}$$

The choice of the scaling mass, $M_1 \in \boldsymbol{R}^{n\times n}$, may be chosen to reflect a desired type of uncertainty representation. For example, Eq. (4) is the standard equation for multiplicative uncertainty if $M_1 = M_0$. Alternatively, Eq. (4) is the standard equation for additive uncertainty if $M_1 = 1$.

The weighting matrix, $W_M \in \boldsymbol{R}^{n\times n}$, is typically used to note the size of the uncertainty. The actual uncertainty operator is not allowed to have a norm greater than unity and so the total error in the parameter is determined by W_M. For example, if $M_1 = M_0$ and $W_M = 0.1$, then the mass is considered to have 10% uncertainty.

The damping matrix of the structural dynamics can also have errors. Introduce a weighted uncertainty operator, $\Delta_C \in \boldsymbol{R}^{n\times n}$, to the nominal matrix, $C_0 \in \boldsymbol{R}^{n\times n}$, as shown in Eq. (5). There are real, constant scaling matrices given by $C_1 \in \boldsymbol{R}^{n\times n}$ and $W_C \in \boldsymbol{R}^{n\times n}$:

$$C = C_0 + C_1 W_C \Delta_C \tag{5}$$

The stiffness matrix is another source of error in the equation of motion. Introduce $\Delta_K \in \boldsymbol{R}^{n\times n}$ as the uncertainty operator to associate with stiffness. There are also real and constant scaling matrices for the uncertainty that are given by $K_1 \in \boldsymbol{R}^{n\times n}$ and $W_K \in \boldsymbol{R}^{n\times n}$:

$$K = K_0 + K_1 W_K \Delta_K \tag{6}$$

The terms in the aerodynamic forces are also obvious choices for associating parametric uncertainty. Specifically, these lag terms have a pole that is a real scalar that is often difficult to compute and may be in error. Equation (7) presents the formulation for uncertainty in the first lag term. The nominal pole β_{1_0} is affected by the operator

$\Delta_\beta \in \boldsymbol{R}$. The weightings, $\beta_{1_1} \in \boldsymbol{R}$ and $W_{\beta_1} \in \boldsymbol{R}$, are again introduced to scale the size of the uncertainty,

$$\beta_1 = \beta_{1_0} + \beta_{1_1} W_{\beta_1} \Delta_{\beta_1} \tag{7}$$

Model Formulation

Parameterization Around Airspeed

The basic concept of the μ-method analysis is to find a worst-case perturbation to a flight condition that results in the onset of flutter. This basic concept is used to develop the original nonmatch-point formulation and the new match-point formulation presented in this paper; however, the flight condition of interest is changed between the formulations. The original formulation considers a perturbation to dynamic pressure, whereas the new formulation considers a perturbation to airspeed.

Assume that the model is formulated to describe the aeroelastic dynamics at a true airspeed of V_0. Consider a perturbation $\delta_V \in \boldsymbol{R}$ that affects this nominal airspeed. The airspeed used in the model is then a simple scalar addition,

$$V = V_0 + \delta_V \tag{8}$$

Introducing the expression of Eq. (8) into the equation of motion is not sufficient to guarantee a match-point solution. The equation of motion in Eq. (3) demonstrates that the system, even at constant Mach, depends on separate parameters that describe flight condition. These parameters are the airspeed and the dynamic pressure. Alternatively, the dynamic pressure can be expressed such that the equation of motion depends on airspeed and density.

The key to computing match-point flutter solutions using μ-method analysis is to formulate the model as a function of a single parameter that describes flight condition. This formulation is accomplished by replacing the dependence on density with an equivalent dependence on airspeed.

A function is introduced to approximate the density. This function is necessary because there is no closed-form solution that relates density to airspeed for a wide range of values in a standard atmosphere. There are many polynomials that may be used; however, a third-order polynomial as given in Eq. (9) has been shown to provide an excellent approximation for most values of density and airspeed:

$$\rho \approx p_0 + p_1 V + p_2 V^2 + p_3 V^3 \tag{9}$$

The value of density for any airspeed is determined by parameters $p_0, p_1, p_2, p_3 \in \boldsymbol{R}$ that are the coefficients in the polynomial. These parameters are chosen by a least-squares fit of the polynomial to known match-point atmospheric data. The density in the equation of motion is simply replaced with the function in Eq. (9) so that the resulting dynamics are expressed as a function of airspeed only.

Match-point flutter speeds are computed using the μ-method technique by considering the worst-case value of δ_V. The result is inherently a match-point solution because the flight condition is entirely determined by δ_V. Thus, the computed flutter speed is independent of the choice of V_0 and so no iterations are required to compute a match-point solution.

A match-point formulation can be derived by considering a perturbation to a different parameter such as dynamic pressure. That formulation would result by replacing the airspeed parameter with an equivalent function that is dependent on dynamic pressure. The algebra presented considers a perturbation to airspeed; however, it is straightforward to derive the model with a perturbation to dynamic pressure. The match-point flutter speed is easily converted to a match-point flutter pressure so that there is no need to derive and present multiple formulations of the model.

Subsystems of the Equation of Motion

The equation of motion for a general aeroelastic system is written as a relationship of structural and aerodynamic forces as expressed in Eq. (3). It is convenient to introduce parameters to replace some expressions in the equation to motion. These parameters allow the aeroelastic system to be expressed as several interconnected subsystems. Specifically, there are subsystems that can be written to represent the structure, the dynamic pressure, the quasi-steady aerodynamics, and the lag terms of the unsteady aerodynamics.

Introduce a parameter $\boldsymbol{q}_0$ to represent the force that is contributed by the quasi-steady aerodynamics. This parameter replaces the expression that includes the steady component of the aerodynamics and the equivalent damping and inertia associated with the quasi-steady portion of the aerodynamic forces,

$$\boldsymbol{q}_0 = \left[A_0 + A_1(b/V)s + A_2(b^2/V^2)s^2\right]\eta \tag{10}$$

Parameters are also defined to replace the lag terms. These parameters $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are individually defined such that a separate parameter is associated with a separate lag term:

$$\boldsymbol{q}_1 = A_3 \frac{(b/V)s}{(b/V)s + \beta_1}\eta \tag{11}$$

$$\boldsymbol{q}_2 = A_4 \frac{(b/V)s}{(b/V)s + \beta_2}\eta \tag{12}$$

The total aerodynamic forces that act on the system are a summation of the quasi-steady and unsteady effects. Define a new parameter $\boldsymbol{q}$ to represent these total forces,

$$\boldsymbol{q} = \boldsymbol{q}_0 + \boldsymbol{q}_1 + \boldsymbol{q}_2 \tag{13}$$

The form of Eq. (3) is written to separate the structural and aerodynamic effects. A parameter $\boldsymbol{a}$ is defined to represent the contribution of the aerodynamic effects. This parameter is defined by scaling the aerodynamic forces, given by $\boldsymbol{q}$ in Eq. (13), with the dynamic pressure,

$$\boldsymbol{a} = \bar{q}\boldsymbol{q} \tag{14}$$

The equation of motion is now written by relating the structural dynamics to the parameter that represents the aerodynamic forces. This relationship is demonstrated by

$$M\ddot{\eta} + C\dot{\eta} + K\eta = -\boldsymbol{a} \tag{15}$$

The procedure that is used to formulate the aeroelastic system in the μ-method framework is to consider each subsystem. A model is generated that describes a particular subsystem as a linear fractional transformation (LFT). The LFT will relate the nominal dynamics of the subsystem with any velocity perturbations or uncertainties. The total system is formulated by combining each LFT using standard LFT operations. The result is a system that is represented as an LFT between the total nominal dynamics and a structured operator that contains all perturbations and uncertainties.

Structural Dynamics

Consider the equation for the structural dynamics from Eq. (15) that relates the mass, stiffness, and damping matrices to the aerodynamic forces. Each of the structural matrices is a potential source of modeling error. This modeling error can be represented by uncertainty operators. Specifically, the error is introduced in the expressions for the mass matrix in Eq. (4), the damping matrix in Eq. (5), and the stiffness matrix in Eq. (6). These expressions replace the general matrices with relationships between nominal matrices and their associated uncertainty operators.

Introduce error in the structural dynamics by substituting the uncertain expressions for the general matrices. The equation of motion for this subsystem is then expanded to separate the uncertain terms from the nominal dynamics:

$$\begin{aligned}
-\boldsymbol{a} &= M\ddot{\eta} + C\dot{\eta} + K\eta = (M_0 + M_1 W_M \Delta_M)\ddot{\eta} \\
&\quad + (C_0 + C_1 W_C \Delta_C)\dot{\eta} + (K_0 + K_1 W_K \Delta_K)\eta = M_0\ddot{\eta} + C_0\dot{\eta} \\
&\quad + K_0\eta + \Delta_M(M_1 W_M \ddot{\eta}) + \Delta_C(C_1 W_C \dot{\eta}) + \Delta_K(K_1 W_K \eta) \\
&= M_0\ddot{\eta} + C_0\dot{\eta} + K_0\eta + \Delta_M z_M + \Delta_C z_C + \Delta_K z_K \\
&= M_0\ddot{\eta} + C_0\dot{\eta} + K_0\eta + w_M + w_C + w_K
\end{aligned} \tag{16}$$

The equation of motion given in Eq. (16) is suitable for representation in the μ-method framework. The equation has removed the explicit dependence on the uncertainty operators with an implicit dependence through feedback. This removal is accomplished by introducing signals to represent the uncertainty operators. The output signals z_M, z_C, and z_K are related to the input signals as $w_M = \Delta_M z_M$, $w_C = \Delta_C z_C$, and $w_K = \Delta_K z_K$. The resulting system is a linear fractional transformation that makes use of several inputs and outputs to relate the nominal dynamics with the uncertainties.

A state-space model of the equation of motion is formulated to derive a model in the μ-method framework. The state-space model is expressed in terms of a matrix quadruple. This model S and its associated quadruple $\{S_A, S_B, S_C, S_D\}$ are derived by analyzing Eq. (16).

The expression for the state matrix S_A is expressed in terms of the nominal dynamics:

$$S_A = \begin{bmatrix} 0 & I \\ -M_0^{-1} K_0 & -M_0^{-1} C_0 \end{bmatrix} \tag{17}$$

The input matrix S_B is also expressed in terms of the nominal dynamics. There are four columns in this matrix to note the relationship of the state derivatives to the three inputs from the uncertainty relationship and the one input from the aerodynamic forces:

$$S_B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -M_0^{-1} & -M_0^{-1} & -M_0^{-1} & -M_0^{-1} \end{bmatrix} \tag{18}$$

The output matrix S_C presents the contributions from the states to the system outputs. These outputs are the three feedbacks to the uncertainty operators and the three measurements of the position, velocity, and acceleration of the modal displacements:

$$S_C = \begin{bmatrix} -M_1 W_M M_0^{-1} K_0 & -M_1 W_M M_0^{-1} C_0 \\ 0 & C_1 W_C \\ K_1 W_K & 0 \\ I & 0 \\ 0 & I \\ -M_0^{-1} K_0 & -M_0^{-1} C_0 \end{bmatrix} \tag{19}$$

The remaining matrix S_D is the feedthrough matrix. This matrix presents the contributions from the inputs to the outputs:

$$S_D = \begin{bmatrix} -M_1 W_M M_0^{-1} & -M_1 W_M M_0^{-1} & -M_1 W_M M_0^{-1} & -M_1 W_M M_0^{-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -M_0^{-1} & -M_0^{-1} & -M_0^{-1} & -M_0^{-1} \end{bmatrix} \tag{20}$$

The system model S is related to the uncertainty operators by a feedback relationship. This relationship is expressed graphically in Fig. 1.

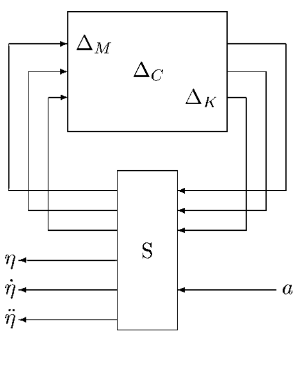

Fig. 1 Subsystem that represents the structural dynamics.

Dynamic Pressure

The structural dynamics and the aerodynamics are related by the dynamic pressure in the original equation of motion of Eq. (1). The match-point formulation requires that the model is expressed in terms of a single parameter for the flight condition. This condition is chosen to be velocity and so the dynamic pressure must be expressed in terms of velocity. The expression uses the approximating function in Eq. (9) to represent the density,

$$\boldsymbol{a} = \tfrac{1}{2}\rho V^2 \boldsymbol{q} = \tfrac{1}{2}\left(p_0 + p_1 V + p_2 V^2 + p_3 V^3\right) V^2 \boldsymbol{q} \tag{21}$$

The expression for the dynamic pressure is simply a combination of velocity and the coefficients in the density polynomial. The coefficients are assumed to be accurate over a given range of velocities; therefore, there are no uncertain variables in this subsystem. The only parameter that must be introduced is a perturbation to airspeed.

The subsystem for dynamic pressure must be expressed as an LFT between the nominal subsystem and the perturbation to velocity. The first step in the derivation of this LFT is to replace the velocity parameter with the relationship between the nominal velocity and its perturbation,

$$\begin{aligned} \boldsymbol{a} &= \tfrac{1}{2}\left(p_0 V^2 + p_1 V^3 + p_2 V^4 + p_3 V^5\right)\boldsymbol{q} = \tfrac{1}{2}\big[p_0 (V_0 + \delta_V)^2 \\ &\quad + p_1 (V_0 + \delta_V)^3 + p_2 (V_0 + \delta_V)^4 + p_3 (V_0 + \delta_V)^5\big]\boldsymbol{q} \\ &= \Big\{\Big[\tfrac{1}{2}\left(p_0 V_0^2 + p_1 V_0^3 + p_2 V_0^4 + p_3 V_0^5\right)\Big] + \Big[\tfrac{1}{2}\big(p_0 2 V_0 \\ &\quad + p_1 3 V_0^2 + p_2 4 V_0^3 + p_3 5 V_0^4\big)\Big]\delta_V + \Big[\tfrac{1}{2}\big(p_0 + p_1 3 V_0 \\ &\quad + p_2 6 V_0^2 + p_3 10 V_0^3\big)\Big]\delta_V^2 + \Big[\tfrac{1}{2}\left(p_1 + p_2 4 V_0 + p_3 10 V_0^2\right)\Big]\delta_V^3 \\ &\quad + \Big[\tfrac{1}{2}(p_2 + p_3 5 V_0)\Big]\delta_V^4 + \left(\tfrac{1}{2} p_3\right)\delta_V^5\Big\}\boldsymbol{q} \end{aligned} \tag{22}$$

The formulation of Eq. (22) separates the nominal terms from the uncertain terms. In particular, the uncertain terms are separated into terms that depend on different orders of the velocity perturbation. Define parameters for the scaling values of each of these separated terms to simplify the presentation:

$$\bar{Q}_0 = \tfrac{1}{2}\left(p_0 V_0^2 + p_1 V_0^3 + p_2 V_0^4 + p_3 V_0^5\right) \tag{23}$$

$$\bar{Q}_1 = \tfrac{1}{2}\left(p_0 2 V_0 + p_1 3 V_0^2 + p_2 4 V_0^3 + p_3 5 V_0^4\right) \tag{24}$$

$$\bar{Q}_2 = \tfrac{1}{2}\left(p_0 + p_1 3 V_0 + p_2 6 V_0^2 + p_3 10 V_0^3\right) \tag{25}$$

$$\bar{Q}_3 = \tfrac{1}{2}\left(p_1 + p_2 4 V_0 + p_3 10 V_0^2\right) \tag{26}$$

$$\bar{Q}_4 = \tfrac{1}{2}\left(p_2 + p_3 5 V_0\right) \tag{27}$$

$$\bar{Q}_5 = \tfrac{1}{2} p_3 \tag{28}$$

Introduce these expressions into Eq. (22) and arrange the groupings of velocity perturbations:

$$\begin{aligned} \boldsymbol{a} &= \left(\bar{Q}_0 + \bar{Q}_1 \delta_V + \bar{Q}_2 \delta_V^2 + \bar{Q}_3 \delta_V^3 + \bar{Q}_4 \delta_V^4 + \bar{Q}_5 \delta_V^5\right)\boldsymbol{q} = \bar{Q}_0 \boldsymbol{q} \\ &\quad + \Big[\bar{Q}_1 + \Big(\bar{Q}_2 + \Big\{\bar{Q}_3 + \big[\bar{Q}_4 + (\bar{Q}_5)\delta_V\big]\delta_V\Big\}\delta_V\Big)\delta_V\Big]\delta_V \boldsymbol{q} \end{aligned} \tag{29}$$

The μ-method framework requires that the explicit dependence of Eq. (29) on the uncertainty perturbations be replaced with an implicit dependence. This replacement is accomplished by introducing fictitious input and output signals that relate the nominal dynamics to the perturbations. This process is essentially identical to that used when formulating the structural subsystem; however, there is an important difference. The dynamic pressure subsystem has terms with nonlinear perturbations, whereas the structural subsystem was entirely linear.

An LFT model, which uses feedback of linear perturbations, can be written to describe some systems with nonlinear dependencies on those perturbations.[4] In particular, an LFT can be expressed for systems that contain polynomials in the perturbations. The standard process for expressing the LFT is to replace the nonlinearities by the introduction of coupled feedback signals.[5,6]

Define $z_1 = \boldsymbol{q}$ and $w_1 = \delta_V z_1$ as a set of feedback signals that depend linearly on the velocity perturbation. These signals are determined by the outer scaling of the parenthetical portion of Eq. (29). Define another set of signals, $z_2 = w_1$ and $w_2 = \delta_V z_2$, that are related by the velocity perturbation. These signals show a linear relationship but actually represent a second-order nonlinearity. This nonlinearity is evidenced by substituting the expression for w_1 into the expression for z_2. Similarly, define $z_3 = w_2$ and $w_3 = \delta_V z_3$ for the cubic nonlinearity, define $z_4 = w_3$ and $w_3 = \delta_V z_4$ to represent the fourth-order nonlinearity, and define $z_5 = w_4$ and $w_5 = \delta_V z_5$ to associate with the fifth-order nonlinearity.

The equation that describes the dynamic pressure subsystem, given in Eq. (29), is now expressed in a form that is suitable for LFT representation. Replace the explicit dependence on the nonlinear perturbations with an implicit dependence that uses feedback signals:

$$\begin{aligned}\boldsymbol{a} &= \bar{Q}_0\boldsymbol{q} + [\bar{Q}_1 + (\bar{Q}_2 + \{\bar{Q}_3 + [\bar{Q}_4 + (\bar{Q}_5)\delta_V]\delta_V\}\delta_V)\delta_V]\delta_V\boldsymbol{q} \\ &= \bar{Q}_0\boldsymbol{q} + \bar{Q}_1 w_1 + \bar{Q}_2 w_2 + \bar{Q}_3 w_3 + \bar{Q}_4 w_4 + \bar{Q}_5 w_5 \end{aligned} \tag{30}$$

The subsystem for dynamic pressure is expressed as a model. This model $\bar{Q}$ is the matrix that relates the inputs w_1–w_5 and $\boldsymbol{q}$ to the outputs z_1–z_5 and $\boldsymbol{a}$, for the system given in Eq. (30). The formulation of $\bar{Q}$ can be expressed using the definitions given in Eqs. (23–28):

$$\bar{Q} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & I \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ \bar{Q}_1 & \bar{Q}_2 & \bar{Q}_3 & \bar{Q}_4 & \bar{Q}_5 & \bar{Q}_0 \end{bmatrix} \tag{31}$$

The model $\bar{Q}$ is related to the velocity perturbation by the LFT shown in Fig. 2. Note that there are $5n$ repeated instances of the velocity perturbation in this LFT. This number is a result of the signals w_1–w_5, each of dimension n, being related to the signals z_1–z_5, also of dimension n, by δ_V.

Quasi-Steady Aerodynamics

A subsystem is also formulated to represent the quasi-steady aerodynamics. This subsystem describes the relationship given in Eq. (10) and equivalently in Eq. (32). The equivalent expression is presented to remove the fractional terms and to simplify the derivations that follow,

$$\begin{aligned}\boldsymbol{q}_0 &= \left[A_0 + (b/V)sA_1 + (b^2/V^2)s^2A_2\right]\eta \\ V^2\boldsymbol{q}_0 &= V^2A_0\eta + VA_1bs\eta + A_2b^2s^2\eta \\ V^2\boldsymbol{q}_0 &= V^2A_0\eta + VA_1b\dot{\eta} + A_2b^2\ddot{\eta}\end{aligned} \tag{32}$$

The expression for the quasi-steady aerodynamics is obviously dependent on the velocity perturbation and, like the expression for dynamic pressure, does not contain any terms with which parametric uncertainty may be associated. The only parameters that could have errors are the reference length b and the A_0, A_1, and A_2 matrices. The reference length is a known quantity from the modeling process and so it will not have any errors. The matrices are determined by computational analysis and so may have errors; however, it is difficult to model these errors as parametric uncertainty. Thus, the modeling of this subsystem only considers the LFT to relate the nominal dynamics to the velocity perturbation.

Introduce the velocity perturbation to Eq. (32) and present an equation that solves for $\boldsymbol{q}_0$:

$$(V_0 + \delta_V)^2\boldsymbol{q}_0 = (V_0 + \delta_V)^2A_0\eta + (V_0 + \delta_V)A_1b\dot{\eta} + A_2b^2\ddot{\eta}$$

$$\left(V_0^2 + 2V_0\delta_V + \delta_V^2\right)\boldsymbol{q}_0 = \left(V_0^2 + 2V_0\delta_V + \delta_V^2\right)A_0\eta + (V_0 + \delta_V)A_1b\dot{\eta} + A_2b^2\ddot{\eta}$$

$$\begin{aligned}\boldsymbol{q}_0 &= \left(A_0\eta + \frac{1}{V_0}A_1b\dot{\eta} + \frac{1}{V_0^2}A_2b^2\ddot{\eta}\right) + \left(2\frac{1}{V_0}A_0\eta + \frac{1}{V_0^2}A_1b\dot{\eta}\right)\delta_V \\ &\quad + \left(\frac{1}{V_0^2}A_0\eta\right)\delta_V^2 - \left(2\frac{1}{V_0} + \frac{1}{V_0^2}\delta_V\right)\delta_V\boldsymbol{q}_0\end{aligned} \tag{33}$$

The expression in Eq. (33) presents the computation of $\boldsymbol{q}_0$ as a nonlinear function of the velocity perturbation. This function is a polynomial dependence on nonlinear perturbation terms and so can be expressed as an LFT. The model formulation defines signals and inputs and outputs that relate the nominal model to the velocity perturbation.

The dynamics demonstrate a linear dependency that is straightforward to rewrite. Define $z_1 = 2(1/V_0)A_0\eta + (1/V_0^2)A_1b\dot{\eta}$ as the output signal and $w_1 = \delta_V z_1$ as the related input signal that replace this linear dependency. There are also nonlinear dependencies that must be removed. The first nonlinearity δ_V^2 is removed by introducing $z_2 = \eta$ and $w_2 = \delta_V z_2$ along with $z_3 = (1/V_0^2)A_0 w_2$ and $w_3 = \delta_V z_3$. The remaining dependency is removed by defining $z_4 = q_0$ and $w_4 = \delta_V z_4$ to represent the linear portion and $z_5 = (1/V_0^2)w_4$ and $w_5 = \delta_V z_5$ to represent the nonlinear portion.

The output q_0 of the quasi-steady aerodynamics subsystem is expressed in terms of these inputs and outputs. Simply replace the explicit dependence on δ_V in Eq. (33) with the definitions of the new signals to derive

$$\begin{aligned}\boldsymbol{q}_0 &= A_0\eta + (1/V_0)A_1b\dot{\eta} + \left(1/V_0^2\right)A_2b^2\ddot{\eta} \\ &\quad + w_1 + w_3 - 2(1/V_0)w_4 - w_5\end{aligned} \tag{34}$$

The subsystem model L_0 is now defined by the system that relates $\boldsymbol{q}_0$ to the modal states and state derivatives. The relationship is expressed as a constant matrix:

$$L_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2(1/V_0)A_0 & \left(1/V_0^2\right)A_1b & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & \left(1/V_0^2\right)A_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & I & -2(1/V_0) & -I & A_0 & (1/V_0)A_1b & \left(1/V_0^2\right)A_2b^2 \\ 0 & 0 & 0 & \left(1/V_0^2\right) & 0 & 0 & 0 & 0 \\ I & 0 & I & -2(1/V_0) & -I & A_0 & (1/V_0)A_1b & \left(1/V_0^2\right)A_2b^2 \end{bmatrix} \tag{35}$$

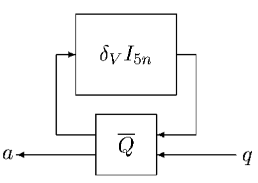

Fig. 2 Subsystem that represents the dynamic pressure.

Figure 3 graphically expresses the subsystem that represents the quasi-steady aerodynamics. This subsystem is now an LFT that relates the nominal dynamics L_0 to the velocity perturbation. Note that this subsystem, similar to the subsystem for dynamic pressure, requires $5n$ instances of the velocity perturbation to account properly for the nonlinear dependence of the subsystem on airspeed.

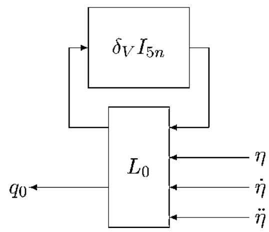

Fig. 3 Subsystem that represents the quasi-steady aerodynamics.

Aerodynamic Lag

The formulation used in this paper considers two lag terms in the unsteady aerodynamics; however, they are of similar form. A general formulation is developed that is applicable to any lag term of this form. Consider the general formulation for the ith lag term.

$$\boldsymbol{q}_i = A_{2+i}\frac{(b/V)s}{(b/V)s + \beta_i}\eta \tag{36}$$

The analysis of the lag dynamics is simplified by defining a signal and introducing it to Eq. (36). This signal x is composed of the dynamic portion of the lag expression,

$$\boldsymbol{q}_i = x \tag{37}$$

The expression for x can be simplified by removing the fractional nature and removing the terms in s,

$$x = \frac{bs}{bs + V\beta_i}\eta, \qquad (bs + V\beta_i)x = bs\eta$$

$$b\dot{x} + \beta_i V x = b\dot{\eta}, \qquad \dot{x} = -\frac{1}{b}V\beta_i x + \dot{\eta} \tag{38}$$

Equation (38) demonstrates that x is actually a state of the lag subsystem. Thus, the output of the subsystem requires both Eqs. (37) and (38).

An LFT model of the lag subsystem requires that the nominal dynamics have only an implicit dependency on any uncertainties or perturbations through feedback. There are two parameters that appear as explicit dependencies in Eq. (38). These parameters are δ_V and δ_{β_i} and represent the velocity perturbation and the error in the lag pole. Expand the expression for $\dot{x}$ to note these dependencies and replace them with feedback signals:

$$\begin{aligned}
\dot{x} &= -\frac{1}{b}V\beta x + \dot{\eta} = -\frac{1}{b}(V_0 + \delta_V)\big(\beta_{i_0} + \beta_{i_1}W_{\beta_i}\Delta_{\beta_i}\big)x + \dot{\eta} \\
&= -\frac{1}{b}V_0\beta_{i_0}x - \frac{1}{b}\beta_{i_0}\delta_V x - \frac{1}{b}\beta_{i_1}W_{\beta_i}\Delta_{\beta_i}\delta_V x \\
&\quad - \frac{1}{b}V_0\beta_{i_1}W_{\beta_i}\Delta_{\beta_i}x + \dot{\eta} = -\frac{1}{b}V_0\beta_{i_0}x - \frac{1}{b}\beta_{i_0}\delta_V z_1 \\
&\quad - \frac{1}{b}\beta_{i_1}W_{\beta_i}\Delta_{\beta_i}\delta_V z_1 - \frac{1}{b}V_0\beta_{i_1}W_{\beta_i}\Delta_{\beta_i}z_3 + \dot{\eta} = -\frac{1}{b}V_0\beta_{i_0}x \\
&\quad - \frac{1}{b}\beta_{i_0}w_1 - \frac{1}{b}\beta_{i_1}W_{\beta_i}\Delta_{\beta_i}w_1 - \frac{1}{b}V_0\beta_{i_1}W_{\beta_i}w_3 + \dot{\eta} = -\frac{1}{b}V_0\beta_{i_0}x \\
&\quad - \frac{1}{b}\beta_{i_0}w_1 - \frac{1}{b}\beta_{i_1}W_{\beta_i}\Delta_{\beta_i}z_2 - \frac{1}{b}V_0\beta_{i_1}W_{\beta_i}w_3 + \dot{\eta} = -\frac{1}{b}V_0\beta_{i_0}x \\
&\quad - \frac{1}{b}\beta_{i_0}w_1 - \frac{1}{b}\beta_{i_1}W_{\beta_i}w_2 - \frac{1}{b}V_0\beta_{i_1}W_{\beta_i}w_3 + \dot{\eta}
\end{aligned} \tag{39}$$

Equation (39) uses several inputs and outputs to express the implicit dependence on uncertainties through feedback. The first set of signals is $z_1 = x$ and $w_1 = \delta_V z_1$ and relates the nominal dynamics to the linear dependency on the velocity perturbation. The next set of signals, $z_2 = w_1$ and $w_2 = \Delta_{\beta_i} z_2$, relates the velocity perturbation to the uncertainty in the lag pole. The last set of signals, $z_3 = x$ and $w_3 = \Delta_{\beta_i} z_3$, relates the nominal dynamics to the linear dependency on the uncertainty in the lag pole.

The subsystem that describes a lag term is expressed in terms of the input and output groups. Specifically, there are three sets of inputs and outputs. Two of the sets are related by feedback and are associated with the velocity perturbation and the uncertainty in the lag pole. The last input and output represents the derivative of the modal displacement and the force from the lag term. Define L_i as the model for the ith lag.

$$L_i = \left[\begin{array}{c|ccccc}
-(1/b)V_0\beta_{i_0} & -(1/b)\beta_{i_0} & -(1/b)\beta_{i_1}W_{\beta_i} & -(1/b)V_0\beta_{i_1}W_{\beta_i} & 1 \\
\hline
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
A_{2+i} & 0 & 0 & 0 & 0
\end{array}\right] \tag{40}$$

The relationship between L_i and the feedback parameters is shown in Fig. 4. Figure 4 shows that the feedback matrix uses only n instances of the velocity perturbation but $2n$ instances of the uncertainty in the lag pole.

The formulation of the subsystems that represent the lag terms is straightforward to accomplish using the definition of L_i. Simply replace each ith variable with its first or second counterpart. For example, replace β_{i_0} with β_{1_0} when formulating the model of L_1 from the definition of L_i.

LFT Formulation

The aeroelastic system is formulated by combining the subsystems together into a single model. The relationships between the subsystems is easily demonstrated graphically. The single model results by combining the structural dynamics in Fig. 1, the dynamic pressure in Fig. 2, the quasi-steady aerodynamics in Fig. 3, and the lag terms of the unsteady aerodynamics in Fig. 4. These systems are related as shown in Fig. 5.

Figure 5 shows a convenient feature that results by considering subsystems, namely, the ability to consider any number of lag terms. The model is formulated for a system with two lags, but it is straightforward to change that number. For example, a system with a single lag results by simply removing L_2 from the model. Also, a system with three lags results by simply adding an additional block that is similar in nature to L_1 and L_2. Thus, the tediousness associated with changing the number of lags in the model formulation is greatly alleviated.

Another convenient feature of the subsystems approach is demonstrated in Fig. 5. This feature is the ability to change easily uncertainty descriptions. Each subsystem is dependent on different elements of the aeroelastic dynamics and so is dependent on different uncertainties. For example, uncertainty parameters associated with stiffness can be introduced or eliminated by simply changing the S subsystem. The remaining subsystems do not need to be analyzed and formulated again. This feature simplifies the process of considering different uncertainty configurations for an aeroelastic system.

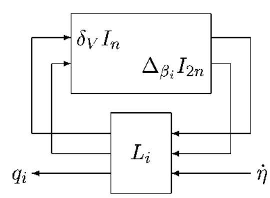

Fig. 4 Subsystem that represents a lag term.

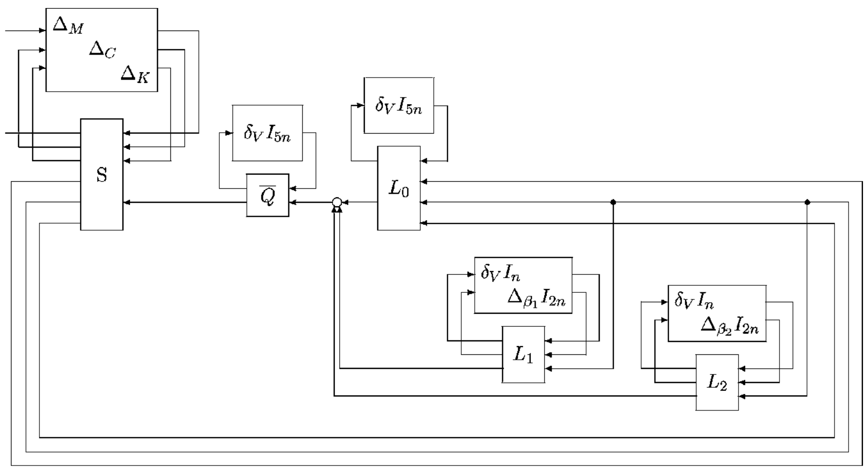

Fig. 5 Aeroelastic model that is formulated by combining subsystems.

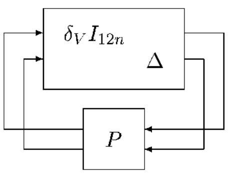

Fig. 6 Model that is used for μ-method analysis.

The subsystem approach is also convenient for model formulation because of the properties of LFTs. Specifically, systems composed of LFT elements can be expressed as a single LFT. This property enables Fig. 5 to be represented by Fig. 6.

The nominal dynamics in Fig. 6 are related to the velocity perturbation and uncertainties through a feedback relationship. In particular, there are $12n$ instances of the velocity perturbation that result from the $5n$ instances associated with the dynamic pressure, the $5n$ instances associated with the quasi-steady aerodynamics, and the n instances from each of the lag terms. The uncertainty matrix Δ is defined as a block-diagonal matrix that contains the parametric uncertainties as the elements:

$$\Delta = \begin{bmatrix} \Delta_M & 0 & 0 & 0 & 0 \\ 0 & \Delta_C & 0 & 0 & 0 \\ 0 & 0 & \Delta_K & 0 & 0 \\ 0 & 0 & 0 & \Delta_{\beta_1} & 0 \\ 0 & 0 & 0 & 0 & \Delta_{\beta_2} \end{bmatrix} \tag{41}$$

It is possible that the descriptions in Figs. 5 and 6 are not of minimal dimensions. This would imply that the number of uncertainty instances may be reduced below the numbers that are shown. Consequently, the computation time for μ-method analysis, which depends on the uncertainty description, may be reduced. There are formal techniques that may be applied to reduce the size of these LFT systems[16]; however, it is not clear that there would be much benefit. An informal analysis of the derivation shows that it is difficult to reduce the size of Δ.

Match-Point Robust Flutter Speeds

Match-point flutter speeds are actually quite straightforward to compute. Indeed, the procedure that is used to analyze the new formulation in this paper with the μ method is exactly the same as the procedure that is presented for the original formulation.[2] The main concept here is not to reformulate the μ-method analysis; rather, it is simply to reformulate the plant model.

The μ method computes flutter speeds by finding the smallest perturbation to airspeed for which the plant is not robustly stable. This analysis requires consideration of both the perturbation to airspeed and the uncertainty in the model formulation. The following theorem is presented to describe the condition that is used to compute a robust flutter speed.

Theorem: Given the plant P derived at a nominal velocity V_0 that is related to a perturbation in velocity δ_V and a set of uncertainty operators $\mathbf{\Delta}$ (as in Fig. 6) define the plant $\bar{P}$ using the scaling W_V:

$$\bar{P} = P \begin{bmatrix} W_V I_{12n} & 0 \\ 0 & I \end{bmatrix}$$

Then $V_{\text{rob}} = V_0 + W_V$ is the least-conservative match-point robust flutter speed if and only if $\mu(\bar{P}) = 1$.

Proof: The proof follows directly from Theorem 8.1.3 of Ref. 20.

The reason that the preceding theorem results in a match-point flutter speed is essentially because of the model formulation. In particular, the dynamic pressure is replaced with a function of velocity. This formulation requires that the density be expressed as a polynomial in terms of velocity; however, the expression can be chosen to approximate closely density for a large range of velocities. The resulting formulation is a function of a single parameter that describes flight condition. Thus, the model, despite any perturbations to that parameter, is always a match-point formulation.

Example

Aerostructures Test Wing

The Aerostructures Test Wing (ATW) structure is being utilized at NASA Dryden Flight Research Center.[17] This wing serves as a testbed for investigating preflight and online methods of predicting the onset of flutter. The ATW is essentially a wing and boom assembly. The system has a span of 18 in. and weighs approximately 2.66 lb.

A model is formulated to represent the ATW dynamics. The model was computed by combining a theoretical mass distribution matrix with frequencies and modes shapes from a ground vibration test. This model was computed using a doublet-lattice approach with the ZAERO code.[18] In particular, rational function approximations of the unsteady aerodynamics are generated using the minimum-state approach.[13] State-space models are formulated using these approximations using both the original nonmatch-point formulation and the new match-point formulation so that solutions from each approach can be compared.

The structural matrices that describe the primary modes of the ATW are computed from the model. These matrices are M to represent the mass matrix, C to represent the damping matrix, and K to represent the stiffness matrix:

$$M = \text{diag}(0.0158, 0.0080, 0.0003) \tag{42}$$

$$C = \text{diag}(0.0068, 0.0079, 0.0010) \tag{43}$$

$$K = \text{diag}(117.1808, 150.4427, 67.9218) \tag{44}$$

The model is used to compute aerodynamic influence coefficients at a Mach 0.8 flight condition. These coefficients are used to determine the matrix coefficients that are needed for Roger's[9] formulation of Eq. (2). A least-squares approach is used to compute a set of optimal matrices with respect to parameters of $b = 0.55$ ft for the reference length and $\beta_1 = 0.1$ and $\beta_2 = 0.5$ for the poles:

$$A_0 = \begin{bmatrix} 0.1562 & 0.4833 & -0.0912 \\ -0.0027 & 0.0504 & -0.0076 \\ -0.0499 & -0.1302 & 0.0357 \end{bmatrix} \tag{45}$$

$$A_1 = \begin{bmatrix} 0.6295 & 0.6959 & -0.1671 \\ 0.1605 & 0.3509 & -0.0507 \\ -0.1527 & -0.1397 & 0.0604 \end{bmatrix} \tag{46}$$

$$A_2 = \begin{bmatrix} 0.2837 & 0.0120 & -0.0418 \\ 0.1743 & 0.0437 & -0.0282 \\ -0.0427 & 0.0090 & 0.0139 \end{bmatrix} \tag{47}$$

$$A_3 = \begin{bmatrix} 0.0007 & -0.0055 & 0.0013 \\ -0.0002 & 0.0018 & -0.0004 \\ -0.0003 & 0.0021 & -0.0005 \end{bmatrix} \tag{48}$$

$$A_4 = \begin{bmatrix} -0.0366 & -0.0986 & 0.0284 \\ -0.0040 & -0.0107 & 0.0031 \\ 0.0086 & 0.0231 & -0.0067 \end{bmatrix} \tag{49}$$

Also, a model is formulated that replaces density with a function of airspeed. The models are only formulated to describe the dynamics at Mach 0.8 and so the corresponding coefficients are used. Note that this approximation is only valid for a range of airspeeds between 830 and 1050 ft/s:

$$\rho = -0.1287 + (4.839 \times 10^{-4})V - (6.1575 \times 10^{-7})V^2 + (2.6675 \times 10^{-10})V^3 \tag{50}$$

Parametric uncertainty is associated with the nominal dynamics to account for potential modeling errors. This uncertainty is restricted to considering errors in the stiffness matrix. In particular, there may be 5% error in the first bending mode, 10% error in the first torsion mode, and 20% error in the second bending mode. A weighting matrix W_K is used to reflect these levels of potential error and normalize the uncertainty for μ-method analysis.

$$W_K = \text{diag}(0.05, 0.10, 0.20) \tag{51}$$

Nonmatch-Point Flutter Speeds

Flutter speeds are computed by applying the μ-method analysis to the original formulation that does not guarantee the resulting solutions are match-point flutter speeds. These speeds are computed and presented to demonstrate the benefits of using the new formulation that is derived in this paper.

A set of models are used to compute nonmatch-point flutter speeds. Each model is formulated with a different value, V_0, of airspeed. The original formulation changes dynamic pressure but does not change airspeed and so there is no guarantee that the analysis will compute a match-point flutter speed. The values of V_0 that are used to formulate models are 893, 860, 838, and 795 ft/s.

Flutter speeds are computed for the ATW models. The true flutter speed V_{vg} is computed by a standard V–g analysis of the nominal model. The application of μ-method analysis to the nonmatch-point formulation results in nominal flutter speeds V_{nom} that consider the assumed dynamics and robust flutter speeds V_{rob} that consider the effect of the stiffness uncertainty. The resulting speeds that correspond to models formulated at different values of V_0 are given in Table 1.

Table 1 Flutter speeds for the nonmatch-point formulation

V_0, ft/s	V_{nom}, ft/s	V_{rob}, ft/s	V_{vg}, ft/s
893	859	836	860
860	860	837	860
838	860	838	860
795	861	839	860

Table 2 Flutter speeds for the match-point formulation

V_0, ft/s	V_{nom}, ft/s	V_{rob}, ft/s	V_{vg}, ft/s
893	859	836	859
859	859	836	859
836	859	836	859
795	859	836	859

Clearly the approach that is used to formulate these models can cause the μ-method analysis to compute results that are not match-point solutions. For example, the model that is formulated using conditions at sea level, $V_0 = 893$ ft/s, generates a nominal flutter speed with an error of 1 ft/s. This error is not overly large; however, it does indicate that the approach can compute nonmatch flutter speeds.

The speeds in Table 1 indicate that it is possible, although not guaranteed, to compute a match-point solution even using the nonmatch-point formulation. The model that is chosen with a nominal flight condition of $V_0 = 860$ ft/s results in a flutter speed of $V_{\text{nom}} = 860$ ft/s. This represents a match-point condition but is, of course, limited to the model that is formulated with this particular value of nominal airspeed. An iterative approach can be used such that a match-point solution is computed despite the initial value of nominal airspeed; however, this approach is computationally expensive and does not provide a guarantee of convergence.

The robust flutter speeds indicate the same basic result as the nominal flutter speeds. Namely, this formulation is able, but not guaranteed, to generate a match-point solution. The only situation that results in an unstable airspeed that matches the airspeed in the model is when $V_0 = 838$ ft/s. The models at other airspeeds do not result in match-point solutions.

Match-Point Flutter Speeds

Flutter speeds are computed by applying the μ-method to the match-point formulation presented in Fig. 6. Both nominal and robust flutter speeds are computed to allow a complete comparison between the nonmatch-point formulation and the match-point formulation.

There are several models that are used for the robust analysis. These models are formulated at the same values of V_0 that are used in the models from the nonmatch-point formulation. The purpose of using the models at these airspeeds is to demonstrate that the match-point formulation does not depend on the nominal airspeed.

Table 2 presents the flutter speeds that are computed using μ-method analysis for the match-point formulation. The nominal flutter speeds V_{nom} and the robust flutter speeds that consider uncertainty, V_{rob}, are given for the models derived at different values of V_0. Clearly the flutter speeds in Table 2 demonstrate that the new formulation guarantees the computation of match-point solutions. These results differ dramatically from those in Table 1 because the flutter speeds that result from this formulation are now independent of the nominal flight conditions. Note that, as expected, both the nominal and robust speeds are independent of the value of V_0.

Conclusions

The model formulation that is introduced in this paper guarantees that the μ-method analysis computes a match-point flutter speed. This formulation is clearly beneficial in comparison to a previous formulation that requires iterations to compute a match-point solution. The match-point nature of this formulation is a result of the proper treatment of nonlinear uncertainties and perturbations. Also, this formulation is advantageous to previous formulations because of the subsystem approach that allows portions of the model to be changed easily without requiring laborious algebra. The ATW is presented as an example that demonstrates the properties of the new formulation. The μ-method analysis computes match-point solutions for both nominal and robust flutter speeds.

References

[1]Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., *Aeroelasticity*, Addison Wesley, Cambridge, MA, 1955, pp. 527–631.

[2]Lind, R., and Brenner, M., *Robust Aeroservoelastic Stability Analysis*, Springer-Verlag, London, 1999, pp. 55–116.

[3]Packard, A., and Doyle, J., "The Complex Structured Singular Value," *Automatica*, Vol. 29, No. 1, 1993, pp. 71–109.

[4]Zhou, K., Doyle, J. C., and Glover, K., *Robust and Optimal Control*, Prentice–Hall, Upper Saddle River, NJ, 1996.

[5]Boukarim, G. E., and Chow, J. H., "Modeling of Nonlinear System Uncertainties Using a Linear Fractional Transformation Approach," *Proceedings of the American Control Conference*, Inst. of Electrical and Electronics Engineers, Piscataway, NJ, 1998, pp. 2973–2979.

[6]Xu, S. J., and Darouach, M., "On the Robustness of Linear Systems with Nonlinear Uncertain Parameters," *Automatica*, Vol. 34, No. 8, 1998, pp. 1005–1008.

[7]Lind, R., and Brenner, M., "Flutterometer: An On-Line Tool to Predict Robust Flutter Margins," *Journal of Aircraft*, Vol. 37, No. 6, 2000, pp. 1105–1112.

[8]Brenner, M., and Lind, R., "Wavelet-Processed Flight Data for Robust Aeroservoelastic Stability Margins," *Journal of Guidance, Control, and Dynamics*, Vol. 21, No. 6, 1998, pp. 823–829.

[9]Roger, K. L., "Airplane Math Modeling Methods for Active Control Design," *Proceedings of the 44th AGARD Structures and Materials Panel*, CP-228, AGARD, 1977, pp. 4.1–4.11.

[10]Vepa, R., "On the Use of Padé Approximants to Represent Unsteady Aerodynamic Loads for Arbitrarily Small Motions of Wings," AIAA Paper 76-17, Jan. 1976.

[11]Suryanarayan, S., Panda, K., and Prakash, B. G., "Extraction of Inherent Aerodynamic Lag Poles for the Time Domain Representation of Modal Unsteady Airloads," AIAA Paper 93-1591, April 1993.

[12]Gupta, K. K., Brenner, M. J., and Voelker, L. S., "Development of an Integrated Aeroservoelastic Analysis Program and Correlation with Test Data," NASA TP-3120, May 1991.

[13]Karpel, M., "Design for the Active Flutter Suppression and Gust Alleviation Using State-Space Aeroelastic Modeling," *Journal of Aircraft*, Vol. 19, No. 3, 1982, pp. 221–227.

[14]Mujumdar, P. M., and Balan, N., "Multiple Order Pole Pure Lag Rational Function Approximations for Unsteady Aerodynamics," *Journal of Aircraft*, Vol. 32, No. 2, 1995, pp. 334–342.

[15]Balan, N., and Mujumdar, P. M., "Minimum-State Approximation: A Pure Lag Approach," *Journal of Aircraft*, Vol. 35, No. 1, 1998, pp. 161–163.

[16]Cockburn, J. C., and Morton, B. G., "Linear Fractional Representations of Uncertain Systems," *Automatica*, Vol. 33, No. 7, 1977, pp. 1263–1271.

[17]Potter, S., and Lind, R., "Developing Uncertainty Models for Robust Flutter Analysis Using Ground Vibration Test Data," AIAA Paper 2001-1585, April 2001.

[18]"ZAERO Users Guide," Zona Technology, Scottsdale, AZ, 2000.