An Approximate Green's Function for Beams and Application to Contact Problems

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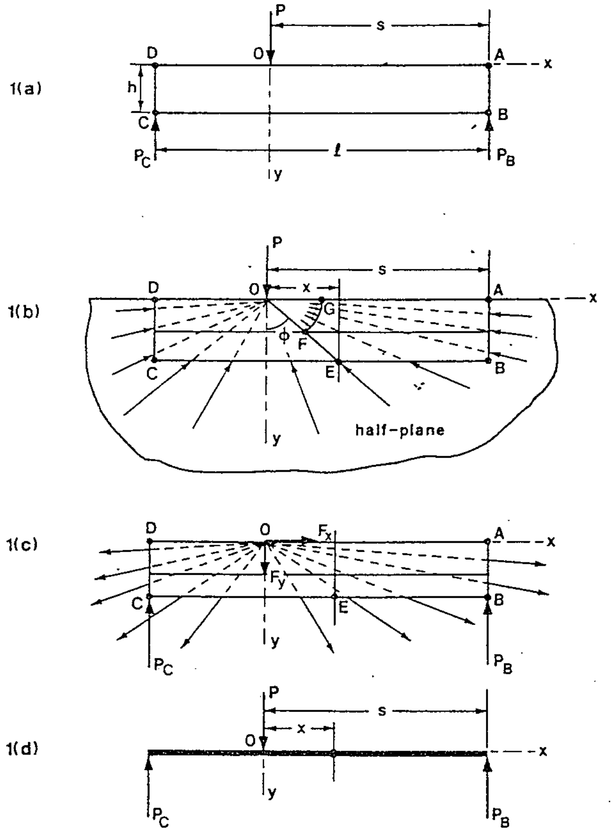

Fig. 1 Principle of superposition for calculating normal displacements on the surface of a beam

Table 1 Comparison of displacement V_{hb} and deflections V_b

x/h	$(EI/Ph^3)v_b$	$(EI/Ph^3)v_{hb}$	$(v_b - v_{hb})/v_{hb}$
0.0	8.5333	8.4612	0.00853
0.1	8.2055	8.1347	0.00869
0.2	7.8624	7.7942	0.00875
0.3	7.5049	7.4401	0.00871
0.4	7.1339	7.0729	0.00862
0.5	6.7500	6.6932	0.00849
0.6	6.3541	6.3016	0.00833
0.7	5.9471	5.8989	0.00816
0.8	5.5296	5.4858	0.00798
0.9	5.1025	5.0630	0.00781
1.0	4.6667	4.6313	0.00764

1 Introduction

The problem of indentation of a beam supported at both ends has been solved by Sankar and Sun (1983) and Keer and Miller (1983). The basic principle in both methods was using the elasticity solution to describe displacements and stresses in the vicinity of contact, and using the beam-theory equations to describe the global behavior. They differed in the numerical technique of solving the contact problem, but reached essentially the same conclusions. It may be noted that the basic information needed to solve a contact problem is the Green's function for surface displacements. Then, the problem may be formulated in terms of an integral equation, which may be solved numerically. Once the contact area and the contact stresses beneath the indenter are found, the stress field in the contacting bodies can be solved by using the equations of elasticity.

In this paper an approximate Green's function for normal displacements on the surface of a beam is proposed. The integral equation for frictionless contact between a beam and a rigid cylindrical indenter is formed in terms of the Green's function. The integral equation is solved by a least squares approximation procedure. A numerical example is given for the problem of central indentation of a simply supported isotropic beam by a smooth, rigid cylinder. The results for the contact stresses beneath the indenter are compared with those given by Keer and Miller (1983).

2 An Approximate Green's Function

Consider the problem of a simply supported beam of rectangular cross section and unit width subjected to a concentrated force P as shown in Fig. 1(a). In the context of contact problems our interest is in determining the surface displacements $v(x, 0)$ in the y direction. As described in Timoshenko and Goodier (1970) the solution to the problem in Fig. 1(a) can be obtained as the superposition of solutions of systems shown in Figs. 1(b) and 1(c). In the above reference such a superposition procedure has been used to calculate stresses in a beam subjected to a concentrated force. We shall extend the same method for determining the displacements as

$$v(x,0) = v_h(x,0) + v_{hb}(x,0), \tag{1}$$

where v_h and v_{hb} are the displacements of systems in Figs. 1(b) and 1(c), respectively.

In Fig. 1(b) the force P acts on a half-plane. The expression for surface displacements in a half-plane of unit thickness under plane stress is given by (Timoshenko and Goodier, 1970)

$$v_h(x,0) = -(2P/\pi E)\ \log\ |x| + \text{constant}. \tag{2}$$

It will be shown later that in contact problems we need only relative displacements, and there is no need to evaluate the constant term.

In Fig. 1(c) radial tensile stresses act on the sides of the rectangular beam supported at the ends. The magnitude of these radial stresses are equal to the magnitude of the compressive stresses on face $ABCD$ in the half-plane in Fig. 1(b). Displacements $v_{hb}(x, 0)$ for the problem shown in Fig. 1(c) can be obtained from the beam theory. It will be further shown that the displacement v_{hb} is approximately equal to the deflection v_b in a beam subjected to a concentrated force P as shown in Fig. 1(d).

The bending moment about the centroid at any section of the beam in Fig. 1(c) can be easily computed if we note that the radial tractions on face ABE are statically equivalent to the radial pressure over the circular arc FG of an arbitrary radius r in Fig. 1(b). The expression for this radial pressure distribution is given in Timoshenko and Goodier (1970) as

$$\sigma_{rr} = -(2P\cos\theta)/\pi r, \tag{3}$$

where θ is measured in a counter-clockwise sense from the y axis. The resultant of this radial pressure is equivalent to two forces F_x and F_y (Fig. 1(c)) acting at O given by

$$F_x = P(1+\cos 2\phi)/2\pi$$

and (4)

$$F_y = P(\pi - 2\phi - \sin 2\phi)/2\pi,$$

where $\tan\phi = x/h$.

The bending moment $M(x)$ at any section can be calculated as

$$M(x) = \lambda P(s-x) - F_x(h/2) + F_y x, \tag{5}$$

where $\lambda = 1 - s/\ell$.

The displacement v_{hb} is then obtained by integrating $M(x)$ twice as follows:

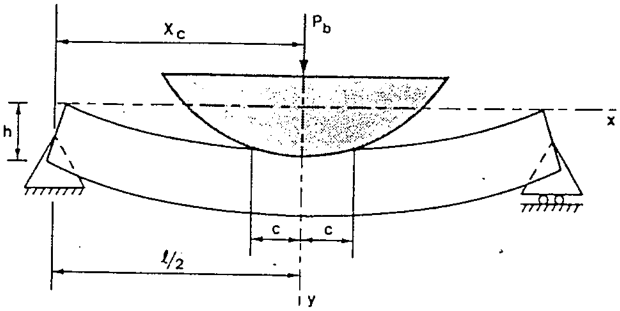

Fig. 2 Central indentation of a simply supported beam

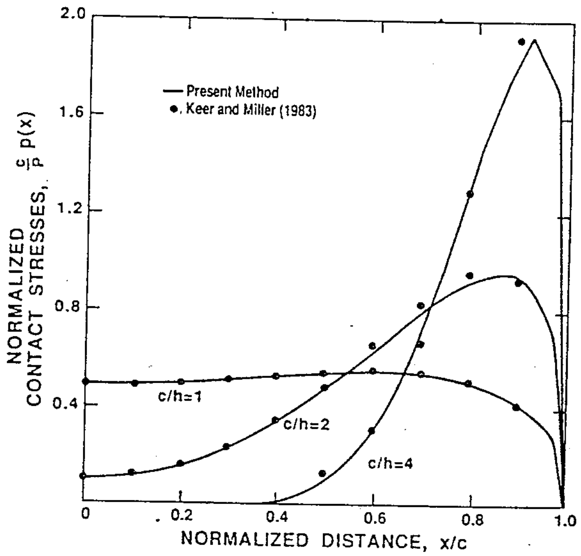

Fig. 3 Contact stress variation in a simply supported beam

$$(EI/Ph^3)V_{hb}(x) = \bar{x}^2\left(\frac{1}{6\pi} - \frac{\lambda s}{2h}\right) + \left(\frac{\lambda - 0.5}{6}\right)\bar{x}^3 + \frac{1}{12\pi}[2\bar{x}^3 \tan^{-1}\bar{x} + \log(1+\bar{x}^2)] + c_1\bar{x} + c_2, \quad (6)$$

where $\bar{x} = x/h$.

In equation (6) E is the Young's modulus, I is the moment of inertia of beam-cross section, and s is the distance of force P from the right support. The constants of integration c_1 and c_2 can be determined from the conditions that displacements are zero at $x = s$ and $x = (s - \ell)$.

The displacements thus obtained were compared with the deflection $v_b(x)$ of a beam subjected to a concentrated force P calculated using the elementary beam formula. The agreement was excellent for large ℓ/h ratios, and also when the load was not very close to either support. The comparison for a worst case ($\ell/h = 10$ and $s/\ell = 0.2$) is given in Table 1. It may be noted that the maximum difference in displacements occurs near the point of application of the concentrated force, but it is still less than 1 percent. An interpretation of this agreement can be that the resultant of F_x and F_y in Fig. 1(c) pass very close to the centroid at the section through E, and does not contribute much to the bending moment, making $M(x)$ approximately equal to $\lambda P(s - x)$, which is identical to the problem in Fig. 1(d). It should be remembered that v_{hb} or v_b is only part of the solution to which v_h has to be added.

In conclusion, it has been shown that the transverse displacements on the surface of a beam due to a concentrated force can be calculated by superposing the displacements on the surface of the half-plane and the beam-theory deflections. The solution for a concentrated force can then be used as a Green's function for computing displacements due to any other type of loading on the beam.

3 Central Indentation of a Simply Supported Beam

Consider a simply supported beam of length ℓ and thickness h (Fig. 2). The beam is assumed to be of unit width and is in a state of plane stress parallel to the $x - y$ plane. The beam is indented by a rigid cylindrical indenter with a parabolic profile given by $y = -x^2/2R$. The contact is assumed to be smooth. Our interest is in determining the contact stress distribution $p(x) = -\sigma_{yy}(x, 0)$ beneath the indenter.

We start with a known contact length $2c$ symmetrical about the center of the beam. The contact length is divided into N_d number of divisions. The unknown contact stresses are assumed to be uniform over each division, that is, over the jth division $p(x) = p_j$. The p_j's are determined from the condition that the deformed shape of the beam in the contact region should conform to the shape of the indenter. This is achieved by choosing N_c number of collocation points x_i's including $|x| = c$, and requiring that

$$v(x_r,0) - v(x_i,0) = (x_i^2 - x_r^2)/2R, \ i = 1,N_c, \quad (7)$$

where x_r is a reference point in the contact region. The left-hand side of equation (7) can be found as a linear function of p_j's using the superposition principle described in Section 2. Thus, the system of equation (7) can be written in the form

$$\sum_j A_{ij}p_j = (x_i^2 - x_r^2)/2R, \ i = 1,N_c. \quad (8)$$

The number N_c has to be at least equal to N_d, but it was found that the variation of contact stresses would be smooth if $N_c > N_d$, and the least squares solution procedure was used to solve for the p_j's. In the numerical examples N_c was equal to 25 and N_d was equal to 20. The IMSL subroutine LLSQF was used in a VAX-11/780 computer to solve the system of linear equations.

In order to compute A_{ij}'s, the solution for relative normal displacements on the boundary of the half-plane due to a uniform load, say p over $-t < x < t$, is needed. For the case of plane stress the relative displacements can be expressed as (Timoshenko and Goodier, 1970)

$$v_h(x,0) - v_h(0,0) = (-2pt/\pi E)[(1-\bar{x})\log|1-\bar{x}| + (1+\bar{x})\log|1+\bar{x}|], \quad (9)$$

where $\bar{x} = x/t$.

The contact stress distribution shown in Fig. 3 corresponds to the case $\ell = 50.8$ mm, $h = 2.54$ mm, $R = 25.4$ mm, and $E = 6.8971$ GPa (10^6 psi). The results agree well with those given in Sankar and Sun (1983). In Fig. 3 the symbols represent the results obtained by Keer and Miller (1983). Again, the agreement is quite good.

4 Summary

The normal displacements on the surface of a beam can be obtained by superposing the beam-theory deflections and the corresponding half-plane solutions. The restriction is that the load should not be very close to a support. This method of superposition simplifies the formulation of the problem of contact between a beam and a rigid indenter. Although the numerical example was concerned with the indentation of a simply supported beam, this method can be easily extended to other boundary conditions, and also to the case of asymmetrical indentation, where the contact stresses may not be symmetrical about the indenter.

References

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