A BEAM THEORY FOR LAMINATED COMPOSITES AND APPLICATION TO TORSION PROBLEMS

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Abstract: *A beam theory for laminated composite beams is derived form the shear deformable laminated plate theory. The displacement field in the beam is derived by retaining the first order terms in the Taylor series expansion for the plate midplane deformations in the width coordinate. The displacements in the beam are expressed in terms of three deflections, three rotations, and one warping term. The equilibrium equations are assumed to be satisfied in an average sense over the width of the beam. This introduces a new set of force and moment resultants for the beam. The principle of minimum potential energy is applied to derive the equilibrium equations and natural boundary conditions. The solution procedure is indicated for the case of a cantilever beam subjected to end loads. A closed form solution is derived for the problem of torsion of a specially orthotropic laminated beam.*

Introduction

Until recently advanced fiber composites were used largely in high performance aircraft structures where weight savings were critical and cost was not an important factor. However development of new fiber and matrix materials, and advances in manufacturing processes have made it possible to use fiber composites in a variety of commercial applications.

For example, fiber composites are used in automobile, boat and aerospace structures, robots and in biomedical devices. They are used not only as laminated plates, but in various shapes and forms. Composite beams have become very common in various applications, *eg.*, automobile suspensions and hip prosthesis. With the increasing use of composites there is a need for simple and efficient analysis procedures for beam like structures. Unlike beams of isotropic materials, composite beams may exhibit a strong coupling between extensional, flexural and twisting modes of deformation. Hence the three problems cannot be treated separately as in the case of isotropic beams. Recently torsion of composite laminates have received some attention, (Kurtz and Whitney, 1988) and (Tsai, Daniel and Yaniv, 1990), for the torsional response of a composite beam can be used to experimentally determine the shear moduli of unidirectional fiber composites.

Another important application of a general anisotropic beam theory is in analyzing delaminations. When a symmetric beam develops a delamination, the two sublaminates on either side of the plane of delamination will be in general anisotropic. In that case extensional or flexural loading on the beam will introduce twisting of the sublaminates. If the delamination length is sufficiently longer than the thickness, then beam theory can be used to determine the strain energy release rate, *eg.*, (Sankar, 1991).

In this paper a beam theory for laminated composite beams is derived from the shear deformable laminated plate theory. The equilibrium equations are assumed to be satisfied in an average sense over the width of the beam. This introduces a new set of force and moment resultants for the beam. The beam equilibrium equations are derived using the minimum potential energy principle. The solution procedure is indicated for the case of a cantilever beam subjected to end loads. The torsion problem for a specially orthotropic laminated beam is derived in closed form and compared with some existing solutions. In addition to the effect of shear deformation, the present solution to torsion problem includes the effect of restrained end of the beam.

Review of Shear Deformation Theory for Laminated Plates

The Shear Deformation Theory (Whitney and Pagano, 1970) for laminated plates is reviewed for the purpose of completion as well as to introduce some notations that will be used in rest of the article. In the following sections boldface letters represent matrices, a superscript T denotes transpose of a matrix, and a comma denotes differentiation with respect to the subscript variables following the comma. The displacement field in a shear deformable plate is:

$$u(x,y,z) = u_0(x,y) + z\psi_x(x,y) \tag{1}$$

$$v(x,y,z) = v_0(x,y) + z\psi_y(x,y) \tag{2}$$

$$w(x,y,z) = w_0(x,y). \tag{3}$$

The laminate constitutive relation can be written as

$$\mathbf{F} = \mathbf{CE} \tag{4}$$

where **F** is the vector of force and moment resultants, **E** is the vector of midplane deformations, and **C** is the laminate stiffness matrix:

$$\mathbf{F}^T = [N_x\ N_y\ N_{xy}\ M_x\ M_y\ M_{xy}\ V_y\ V_x] \tag{5}$$

$$\begin{aligned}\mathbf{E}^T &= [\epsilon_{x0}\ \epsilon_{y0}\ \gamma_{xy0}\ \kappa_x\ \kappa_y\ \kappa_{xy}\ \gamma_{yz}\ \gamma_{xz}] \\ &= [u_{0,x}\ v_{0,y}\ (u_{0,y}+v_{0,x})\ \psi_{x,x}\ \psi_{y,y}\ (\psi_{x,y}+\psi_{y,x})\ (\psi_y+w_{,y})\ (\psi_x+w_{,x})]\end{aligned} \tag{6}$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & 0 \\ \mathbf{B} & \mathbf{D} & 0 \\ 0 & 0 & \mathbf{K} \end{bmatrix}. \tag{7}$$

The **A, B,** and **D** are the extensional, coupling, and flexural stiffness matrices respectively (Jones, 1975). Shear correction factors (Whitney and Pagano, 1970) are included in the laminate shear stiffness matrix **K** . The strain energy per unit area of the laminate, Φ_L, can be derived as

$$\Phi_L = \tfrac{1}{2}\, \mathbf{E}^T \mathbf{C} \mathbf{E}. \tag{8}$$

Derivation of a Composite Beam Theory

Consider a laminated composite beam shown in Figure 1. The midplane displacements and rotations in equations (1–3) can be expanded in the form of a Taylor series in y. Retaining only the first order terms in y, we obtain expressions for the midplane deformations as

$$u_0(x,y) = U(x) + yF(x) \tag{9}$$

$$v_0(x,y) = V(x) + yG(x) \tag{10}$$

$$w_0(x,y) = W(x) + y\theta(x) \tag{11}$$

$$\psi_x(x,y) = \phi(x) + y\alpha(x) \tag{12}$$

$$\psi_y(x,y) = \Psi(x) + yH(x). \tag{13}$$

The terms U, V and W are displacements of points on the longitudinal axis of the beam (x–axis). Similarly ϕ and Ψ are rotations along the x–axis. From the above kinematic assumptions the expressions for normal strain ϵ_{yy} and shear strain γ_{yz} take the form

$$\epsilon_{yy} = G + zH \tag{14}$$

$$\gamma_{yz} = \Psi + yH + \theta \tag{15}$$

We will further assume that the normal strain ϵ_{yy} and shear strain γ_{yz} vanish. This can be accomplished by setting $G=0$, $H=0$, and $\Psi=-\theta$. These assumptions, along with the plate theory assumption $\epsilon_{zz}=0$, imply that beam cross sections normal to the x–axis do not undergo any inplane deformations . Substituting the assumed displacement field in (6), the midplane deformations take the form

$$\mathbf{E} = \overline{\mathbf{E}} + y\hat{\mathbf{E}} \tag{16}$$

where

$$\overline{\mathbf{E}}^T = \left[U' \quad 0 \quad (V' + F) \quad \phi' \quad 0 \quad (\alpha - \theta') \quad 0 \quad (\phi + W')\right] \tag{17}$$

$$\hat{\mathbf{E}}^T = \left[F' \quad 0 \quad 0 \quad \alpha' \quad 0 \quad 0 \quad 0 \quad (\alpha + \theta')\right] \tag{18}$$

and a prime denotes differentiation with respect to x. The constitutive relations (4) become

$$\mathbf{F} = \mathbf{C}(\overline{\mathbf{E}} + y\hat{\mathbf{E}}) \cdot \tag{19}$$

A new set of force and moment resultants for the beam are defined as follows:

$$\overline{\mathbf{F}}(x) = \int_{-b/2}^{+b/2} \mathbf{F}(x,y)dy \tag{20}$$

$$\hat{\mathbf{F}}(x) = \int_{-b/2}^{+b/2} y\mathbf{F}(x,y)dy \cdot \tag{21}$$

Then the laminate constitutive relations (19) take the form

$$\overline{\mathbf{F}} = b\mathbf{C}\overline{\mathbf{E}} \tag{22}$$

$$\hat{\mathbf{F}} = \left(\frac{b^3}{12}\right)\mathbf{C}\hat{\mathbf{E}} \cdot \tag{23}$$

In deriving (22) and (23), **C** is assumed to be constant. Explicit forms of the laminate constitutive relations are presented in the following four expressions (24–27) as they will be used in the next section:

$$\begin{Bmatrix} \overline{N}_x \\ \overline{N}_{xy} \\ \overline{M}_x \\ \overline{M}_{xy} \end{Bmatrix} = b \begin{bmatrix} A_{11} & A_{16} & B_{11} & B_{16} \\ A_{16} & A_{66} & B_{16} & B_{66} \\ B_{11} & B_{16} & D_{11} & D_{16} \\ B_{16} & B_{66} & D_{16} & D_{66} \end{bmatrix} \begin{Bmatrix} U' \\ V' + F \\ \phi' \\ \alpha - \theta' \end{Bmatrix} \tag{24}$$

$$\overline{V}_x = b\kappa^2 A_{55}(\phi + W') \tag{25}$$

$$\begin{Bmatrix} \hat{N}_x \\ \hat{M}_x \end{Bmatrix} = \left(\frac{b^3}{12}\right) \begin{bmatrix} A_{11} & B_{11} \\ B_{11} & D_{11} \end{bmatrix} \begin{Bmatrix} F' \\ \alpha' \end{Bmatrix} \tag{26}$$

$$\hat{V}_x = \left(\frac{b^3}{12}\right)\kappa^2 A_{55}(\alpha + \theta') \tag{27}$$

where the bar and hat accents associate the resultants in the obvious way. In equations (25) and (27) κ^2 is the shear correction factor. From equations (8) and (16) the strain energy per unit length of the beam, Φ_B, can be derived as

$$\Phi_B = \int_{-b/2}^{+b/2} \Phi_L \, dy = \left(\frac{1}{2}\right)\left[b\bar{\mathbf{E}}^T\mathbf{C}\bar{\mathbf{E}} + \left(\frac{b^3}{12}\right)\hat{\mathbf{E}}^T\mathbf{C}\hat{\mathbf{E}}\right] \cdot \tag{28}$$

In order to derive the equations of equilibrium for the beam, we apply the principle of minimum potential energy. For the purpose of illustration we will consider only transverse loading, $q(x,y)$, in the z–direction acting on the beam surface. The total potential energy Π is the sum of strain energy in the beam, Φ, and the potential of the external force, χ. The expressions for the energy terms are derived as shown below. Assuming L is the beam length,

$$\Phi = \int_0^L \Phi_B \, dx \tag{29}$$

$$\chi = -\int_0^L \int_{-b/2}^{+b/2} q(x,y)\, w_0(x,y)\, dy\, dx \tag{30}$$

Substituting for $w_0(x,y)$ from equation (11) we obtain

$$\chi = -\int_0^L \{\bar{q}W(x) + \hat{q}\theta(x)\}\, dx \tag{31}$$

where $\bar{q}$ and $\hat{q}$ are given by

$$\bar{q}(x) = \int_{-b/2}^{+b/2} q(x,y)\, dy \tag{32}$$

$$\hat{q} = \int_{-b/2}^{+b/2} yq(x,y)\, dy \cdot \tag{33}$$

Applying the principle of minimum potential energy, ($\delta\Pi = 0$), (Reddy, 1986), we obtain the following equilibrium equations and boundary conditions:

Variable	Equilibrium equations	Natural boundary conditions	
$\delta U:$	$\dfrac{d\overline{N}_x}{dx} = 0$	$\overline{N}_x\ \delta U = 0$	(34)
$\delta V:$	$\dfrac{d\overline{N}_{xy}}{dx} = 0$	$\overline{N}_{xy}\ \delta V = 0$	(35)
$\delta W:$	$\dfrac{d\overline{V}_x}{dx} + \overline{q} = 0$	$\overline{V}_x\ \delta w = 0$	(36)
$\delta F:$	$\overline{N}_{xy} - \dfrac{d\hat{N}_x}{dx} = 0$	$\hat{N}_x\ \delta F = 0$	(37)
$\delta\theta:$	$\dfrac{d}{dx}(\hat{V}_x - \overline{M}_{xy}) + \hat{q} = 0$	$(\hat{V}_x - \overline{M}_{xy})\delta\theta = 0$	(38)
$\delta\phi:$	$\overline{V}_x - \dfrac{d\overline{M}_x}{dx} = 0$	$\overline{M}_x\ \delta\phi = 0$	(39)
$\delta a:$	$\dfrac{d\hat{M}_x}{dx} = \hat{V}_x + \overline{M}_{xy}$	$\hat{M}_x\ \delta a = 0\,.$	(40)

Substituting for the force and moment resultants in the above differential equations of equilibrium in terms of displacement variables from (22) and (23), one can obtain a system of seven ordinary differential equations for the seven unknown functions. In the following section the solution procedure is illustrated for the case of a cantilever beam subjected to end loads.

Cantilever Beam Subjected to End Loads

Consider the case of a cantilevered laminated beam of rectangular cross section subjected to end loads only, *ie.*, $q(x,y)=0$. The first six equilibrium equations (34–39) can be integrated to yield the following expressions for the force and moment resultants:

$$\overline{N}_x = C_1 \tag{41}$$

$$\overline{N}_{xy} = C_2 \tag{42}$$

$$\overline{V}_x = C_3 \tag{43}$$

$$\hat{N}_x = C_2 x + C_4 \tag{44}$$

$$\hat{V}_x - \overline{M}_{xy} = C_5 \tag{45}$$

$$\overline{M}_x = C_3 x + C_6 \cdot \tag{46}$$

The constants C_1 through C_6 can be determined from the forces and couples applied at the end of the beam, $x = L$. It may be noted that C_5 is equal to the torque T applied at the free end. The procedure for solving equation (40) is as follows. The laminate constitutive relations (24) and (27) can be written as

$$\begin{Bmatrix} \overline{N}_x \\ \overline{N}_{xy} \\ \overline{M}_x \\ \hat{V}_x - \overline{M}_{xy} \end{Bmatrix} = b \begin{bmatrix} A_{11} & A_{16} & B_{11} & -B_{16} \\ A_{16} & A_{66} & B_{16} & -B_{66} \\ B_{11} & B_{16} & D_{11} & -D_{16} \\ -B_{16} & -B_{66} & -D_{16} & D'_{66} \end{bmatrix} \begin{Bmatrix} U' \\ V' + F \\ \phi' \\ \theta' \end{Bmatrix} + b \begin{Bmatrix} B_{16} \\ B_{66} \\ D_{16} \\ D'_{55} \end{Bmatrix} \alpha = b\mathbf{S}\mathbf{e} + b\mathbf{S}_1 \alpha \tag{47}$$

$$\hat{V}_x + \overline{M}_{xy} = b\mathbf{S}_1^T \mathbf{e} + bD'_{66}\alpha \tag{48}$$

where

$$D_{55} = \left(\frac{b^2}{12}\right)\kappa^2 A_{55} \tag{49}$$

$$D'_{55} = D_{55} - D_{66}, \quad D'_{66} = D_{66} + D_{55} \cdot \tag{50}$$

We take advantage of the known forms of the force and moment resultants in equations (41)–(46) to rewrite (47) as

$$\mathbf{F}_1 + x\mathbf{F}_2 = b\mathbf{S}\mathbf{e} + b\mathbf{S}_1 \alpha \tag{51}$$

where

$$\mathbf{F}_1^T = [C_1 \ C_2 \ C_6 \ C_5] \tag{52}$$

$$\mathbf{F}_2^T = [0 \ 0 \ C_3 \ 0] \cdot \tag{53}$$

Solving for $\mathbf{e}$ from (51) and substituting in equation (48) we obtain

$$\hat{V}_x + \overline{M}_{xy} = \mathbf{S}_1{}^T\mathbf{S}^{-1}(\mathbf{F}_1 + x\mathbf{F}_2) + (bD'_{66} - b\mathbf{S}_1^T\mathbf{S}^{-1}\mathbf{S}_1)\alpha \cdot \tag{54}$$

Eliminating F' from (26) we obtain

$$\hat{M}_x = \frac{b^3}{12}D_{11}^*\alpha' + \frac{B_{11}}{A_{11}}\hat{N}_x \tag{55}$$

where

$$D_{11}^* = D_{11} - \frac{B_{11}^2}{A_{11}} \cdot \tag{56}$$

Differentiating (55) with respect to x and using equilibrium equation (37), we obtain

$$\frac{d\hat{M}_x}{dx} = \frac{b^3}{12}D_{11}^*\frac{d^2\alpha}{dx^2} + \frac{B_{11}}{A_{11}}\overline{N}_{xy} \cdot \tag{57}$$

Substituting from (54) and (57) into the last equilibrium equation (40), and noting $\overline{N}_{xy} = C_2$, we obtain a differential equation in $\alpha(x)$ as

$$\frac{d^2\alpha}{dx^2} - \lambda^2\alpha = \left(\frac{12}{b^3D_{11}^*}\right)\left(\mathbf{S}_1^T\mathbf{S}^{-1}(\mathbf{F}_1 + x\mathbf{F}_2) - \frac{B_{11}C_2}{A_{11}}\right) \tag{58}$$

where

$$\lambda^2 = \left(\frac{12D_{66}^*}{b^2D_{11}^*}\right) \tag{59}$$

$$D_{66}^* = D'_{66} - \mathbf{S}_1^T\mathbf{S}^{-1}\mathbf{S}_1 \cdot \tag{60}$$

The solution for α can be written as

$$\alpha = C_7\cosh\lambda x + C_8\sinh\lambda x - \left(\frac{12}{b^3\lambda^2D_{11}^*}\right)\left(\mathbf{S}_1^T\mathbf{S}^{-1}(\mathbf{F}_1 + x\mathbf{F}_2) - \frac{B_{11}C_2}{A_{11}}\right) \cdot \tag{61}$$

The constants C_7 and C_8 can be determined from the boundary conditions $\alpha(0) = 0$, and $\hat{M}_x(L) = C_9$, where C_9 is a known applied force at the beam end. The expression for $\hat{M}_x$ is given in (55). The constants are found to be

$$C_7 = \left(\frac{12}{b^3\lambda^2 D_{11}^*}\right)\left(\mathbf{S}_1^T\mathbf{S}^{-1}\mathbf{F}_1 - \frac{B_{11}C_2}{A_{11}}\right) \tag{62}$$

$$C_8 = \left(\frac{1}{\lambda\cosh\lambda L}\right)\left[\left(\frac{1}{D_{11}^*}\right)\left(\frac{12C_9}{b^3} - \frac{B_{11}}{A_{11}}(C_2L + C_4)\right) - C_7\lambda\sinh\lambda L + \left(\frac{12}{b^3\lambda^2 D_{11}^*}\right)\mathbf{S}_1^T\mathbf{S}^{-1}\mathbf{F}_2\right] \tag{63}$$

The next step is to solve for $F(x)$. From the first of the system of equations (26)

$$\frac{dF}{dx} = \left(\frac{12}{b^3 A_{11}}\right)(C_2x + C_4) - \left(\frac{B_{11}}{A_{11}}\right)\frac{d\alpha}{dx}. \tag{64}$$

Integrating (64) we obtain the solution for $F(x)$ as

$$F = \left(\frac{12}{b^3 A_{11}}\right)\left(C_2\frac{x^2}{2} + C_4x\right) - \left(\frac{B_{11}}{A_{11}}\right)\alpha + C_{10}. \tag{65}$$

Since $F(0) = \alpha(0) = 0$, the constant $C_{10} = 0$. The solutions for F and α can be substituted into (47) to obtain expressions for U', V', ϕ' and θ', which then can be integrated. The constants of integration C_{11}–C_{14} which arise can be found from the boundary conditions: $U(0) = V(0) = \phi(0) = \theta(0) = 0$. The algebra is quite cumbersome in the general case and will not be presented in this paper. However the solution for the case of torsion of specially orthotropic beams will be presented in the following section.

Torsion of Specially Orthotropic Laminated beams

For specially orthotropic beams, $B = 0$, $A_{16} = A_{26} = D_{16} = D_{26} = 0$. We assume that the cantilever beam is subjected to an end torque T. Therefore the constants C_1, C_2, C_3, C_4, C_6 and C_9 are all equal to zero, and $C_5 = T$. Substituting for the constants in all expressions derived in the previous section, we obtain the following:

$$F_1 = [0\ 0\ 0\ T]^{\mathrm{T}} \tag{66}$$

$$F_2 = [0\ 0\ 0\ 0]^{\mathrm{T}} \tag{67}$$

$$\mathrm{S}_1^T \mathrm{S}^{-1} \mathrm{S}_1 = \frac{D_{55}^{'2}}{D_{66}^{'}} \tag{68}$$

$$D_{66}^{*} = \frac{4D_{66}D_{55}}{D_{66}^{'}} \tag{69}$$

$$D_{11}^{*} = D_{11} \tag{70}$$

$$\lambda^2 = \frac{48D_{66}D_{55}}{b^2 D_{11} D_{66}^{'}} \tag{71}$$

$$C_7 = \frac{12D_{55}^{'}T}{b^3\lambda^2 D_{11} D_{66}^{'}}$$

$$= \frac{D_{55}^{'}T}{4bD_{55}D_{66}} \tag{72}$$

$$C_8 = -\,C_7 \tanh \lambda L \tag{73}$$

$$\alpha(x) = C_7(\cosh \lambda x - 1) + C_8 \sinh \lambda x$$

$$= \left(\frac{D_{55}^{'}T}{4bD_{55}D_{66}}\right)\left(\cosh \lambda x - 1 - \tanh \lambda L \ \sinh \lambda x\right) \cdot \tag{74}$$

Substituting for $\alpha(x)$ from (74) in (47) and solving for θ', we obtain

$$\theta' = \left(\frac{1}{bD_{66}^{'}}\right)\left(T - bD_{55}^{'}\alpha\right) \cdot \tag{75}$$

Integrating (75) and noting that $\theta(0) = 0$, we obtain the solution for angle of twist in a specially orthotropic laminated beam subjected to an end torque T:

$$\theta(x) = \frac{T}{D_{66}^{'}}x - \left(\frac{D_{55}^{'2}T}{4D_{55}D_{66}D_{66}^{'}}\right)\left(\frac{\sinh \lambda x}{\lambda} - x - \frac{\tanh \lambda L}{\lambda}(\cosh \lambda x - 1)\right) \cdot \tag{76}$$

For the purpose of comparison with available results we will introduce a nondimensional tip rotation Θ defined as

$$\Theta = \frac{4bD_{66}\theta(L)}{TL}. \tag{77}$$

Then from (76) and (77) the solution for the tip rotation takes the form

$$\Theta = 1 + \frac{D_{66}}{D_{55}} - \frac{\left(1 - \frac{D_{66}}{D_{55}}\right)^2}{\left(1 + \frac{D_{66}}{D_{55}}\right)} \frac{\tanh \lambda L}{\lambda L}. \tag{78}$$

In the above result, the first term on the right hand side corresponds to the classical theory solution for isotropic beams. The shear deformation effects are reflected in the second and third terms. The third term represents the effect of the restrained end $x=0$, where warping is prevented, *ie.*,$\alpha(0)=0$. In Figure 2 Θ is plotted as a function of λL for various values of (D_{66}/D_{55}). It may be seen that the restrained end effects are felt only for $\lambda L<10$. Further the restrained end effects are less pronounced as the ratio of the shear stiffness coefficients (D_{66}/D_{55}) increases. The effect of transverse shear flexibility $(1/D_{55})$ is to increase the angle of twist significantly.

We will compare our results with two available results. If we ignore the shear deformation, ie. let $D_{55} \to \infty$ in equation (78), then we obtain

$$\Theta = 1 + \frac{\tanh \lambda L}{\lambda L} \tag{79}$$

which is identical to the result for an isotropic beam (Boresi, Sidebottom, Seely and Smith, 1978). If we ignore the restrained end effects by letting $\lambda L \to \infty$ in result (78), then we obtain

$$\Theta = 1 + \frac{D_{66}}{D_{55}}. \tag{80}$$

The above results can be compared with that of (Tsai, Daniel and Yaniv, 1990) for a 0° unidirectional composite beam. Let us denote their result by Θ_1. In their notation

$$\Theta_1 = \left(1 - \frac{\tanh\beta}{\beta}\right)^{-1} \tag{81}$$

where

$$\beta = \left(\frac{b}{2h}\right)\sqrt{10\frac{G_{13}}{G_{12}}}\,. \tag{82}$$

In the same notation, result (80) obtained in the present study can be written as

$$\Theta = 1 + \frac{3}{\beta^2}\,. \tag{83}$$

In deriving (81) and (83) the shear correction factor κ^2 is assumed to be 5/6. The two results, Θ and Θ_1, are compared for some practical range of $1/\beta$ in Figure 3. The agreement is quite good. The maximum difference is about 11%, which occurs at $1/\beta \simeq 0.288$. It is interesting to note that

$$\lim_{\beta\to 0} \beta^2\left(1 - \frac{\tanh\beta}{\beta}\right)^{-1} = 3 = \lim_{\beta\to 0} \beta^2\left(1 + \frac{3}{\beta^2}\right) \tag{84}$$

so that $(\Theta/\Theta_1) \to 1$ as $1/\beta \to \infty$.

Conclusions

A beam theory for laminated composites has been derived. The displacements in the beam are expressed in terms of three deflections, three rotations and one warping function. The plate equilibrium equations are assumed to be satisfied in an average sense over the width of the beam. This introduces a new set of force and moment resultants for the beam. The solution procedure is indicated for the case of a cantilever beam subjected to end loads. A closed form solution is derived for the problem of torsion of a specially orthotropic laminated beam. The solution takes into account of shear deformation as well as the effect of restrained ends. The result for angle of twist compare well with available solutions. The energy expressions derived in this paper can be used to formulate a laminated beam finite element which can handle complex boundary condtions and loading cases.

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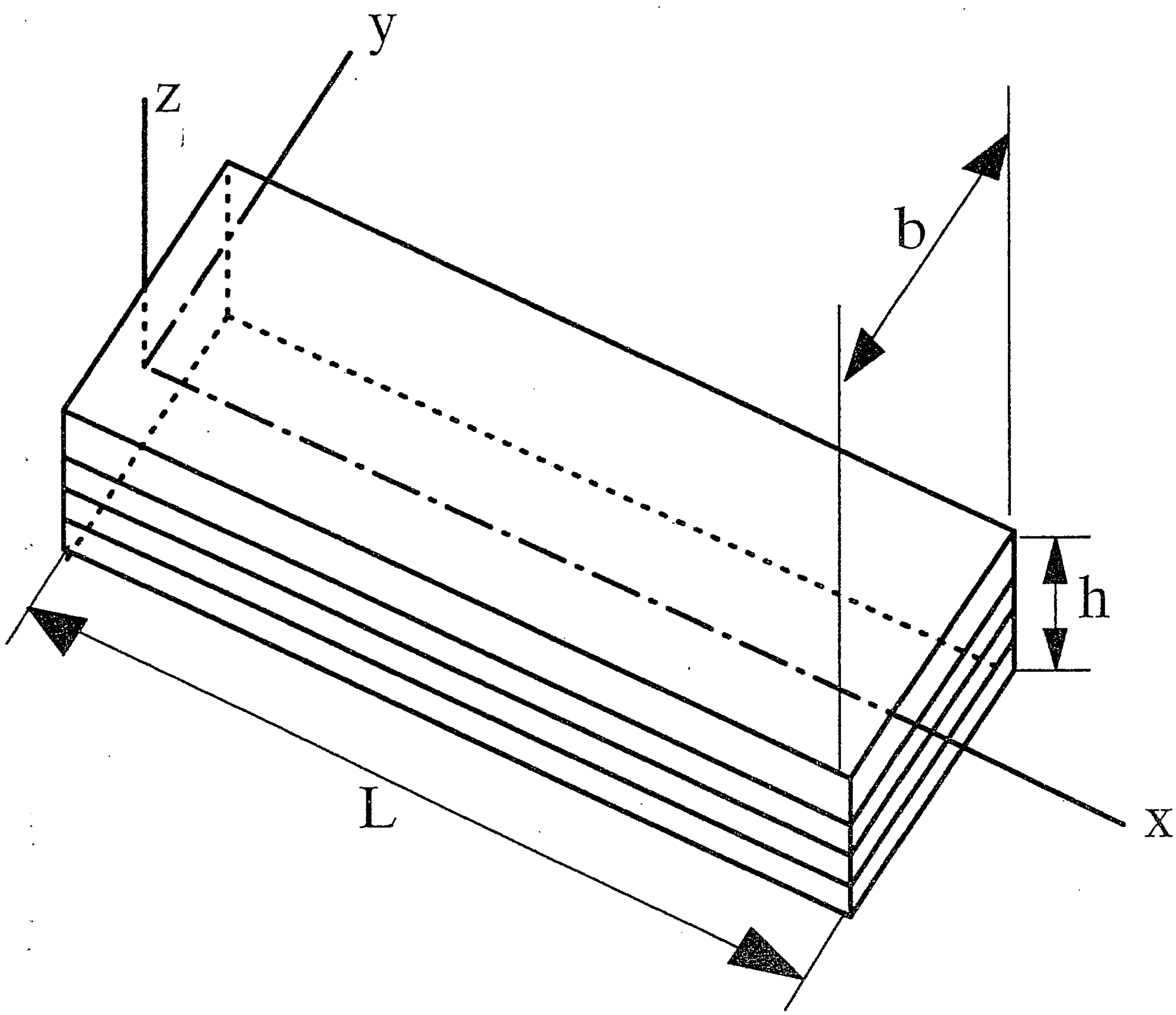

Figure 1. Laminated Composite Beam

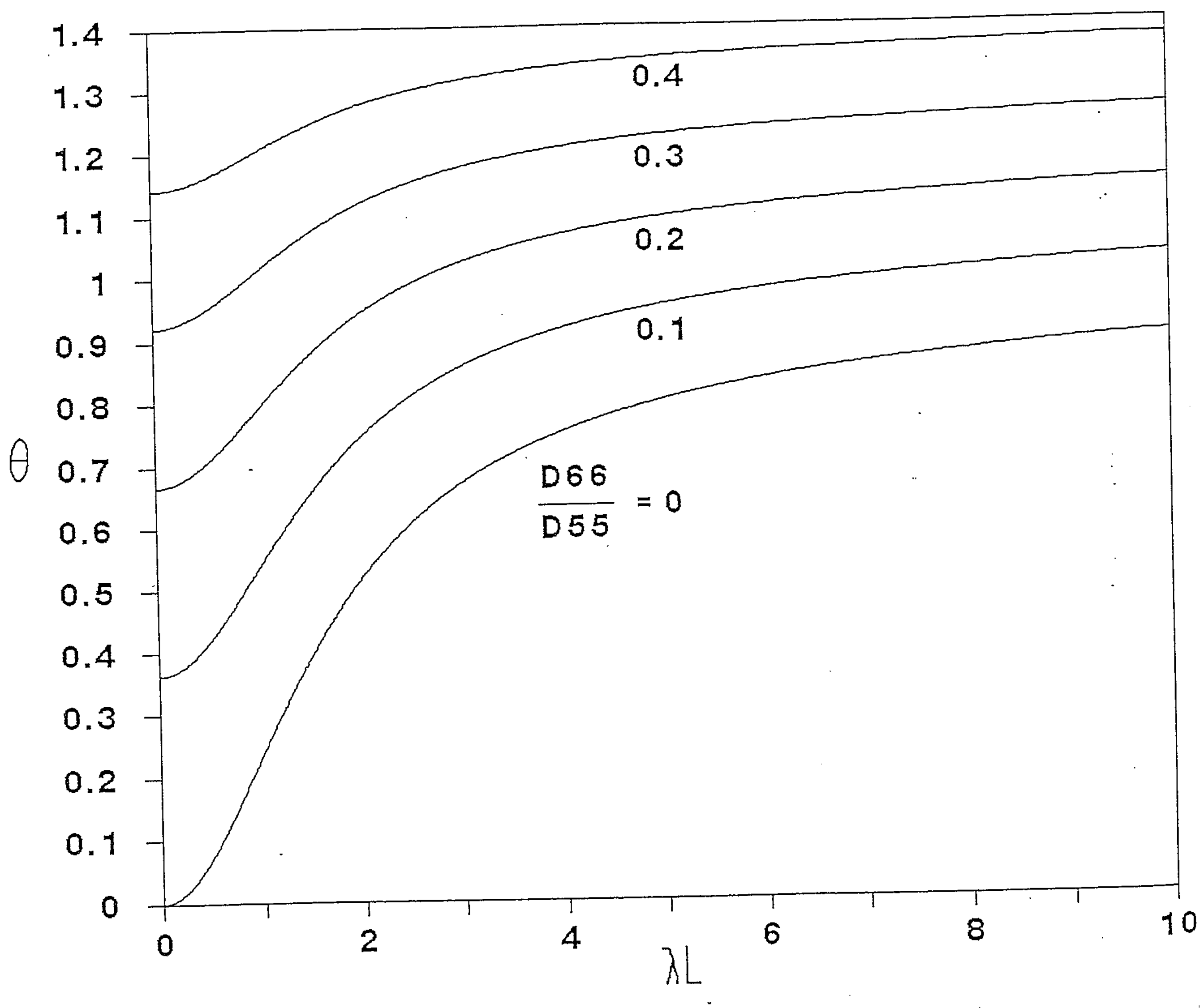

Figure 2. Nondimensional tip rotation Θ (Equation 78)

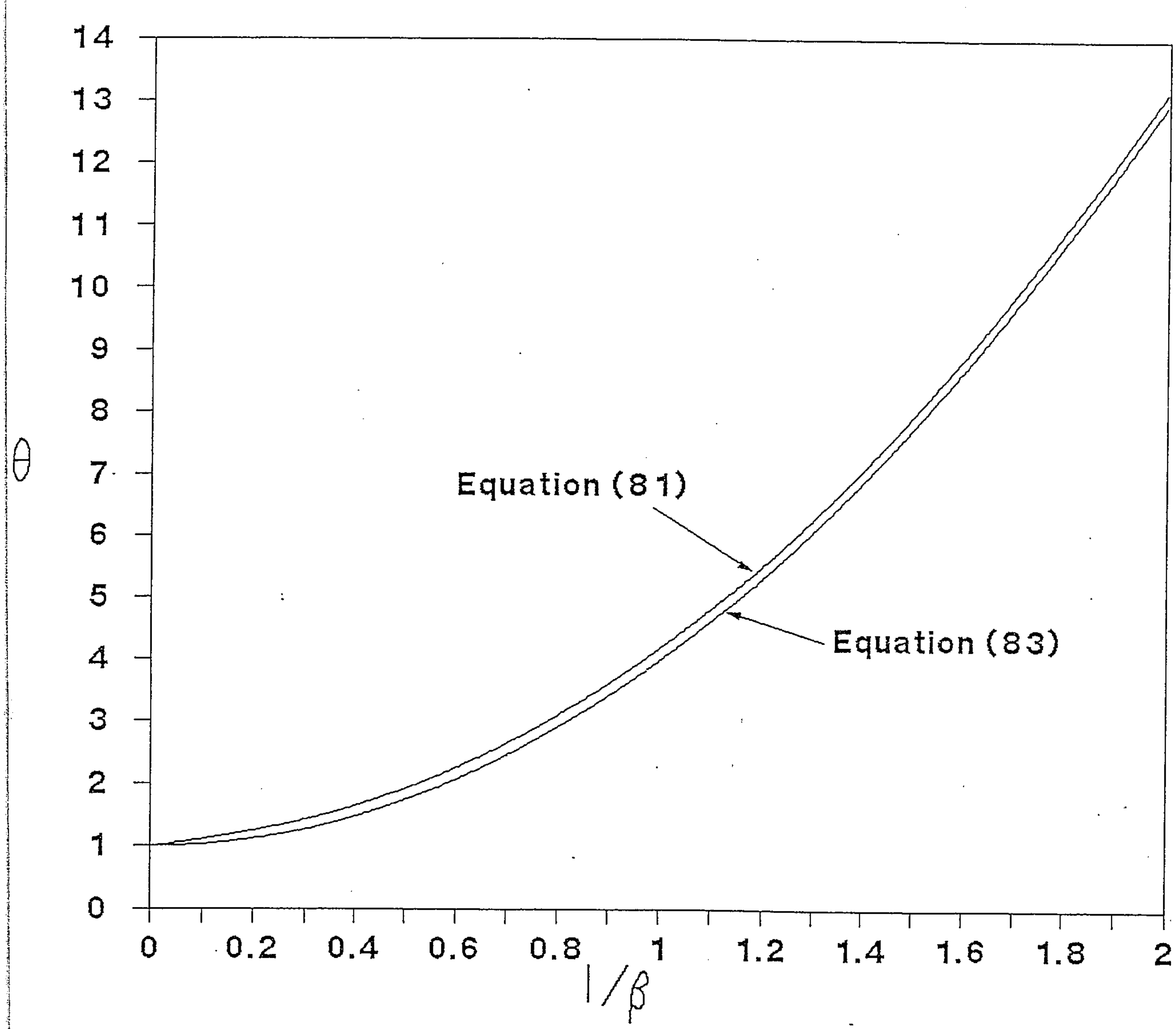

Figure 3. Nondimensional tip rotation Θ for long beams