HEAVISIDE, DIRAC, AND STAIRCASE FUNCTIONS

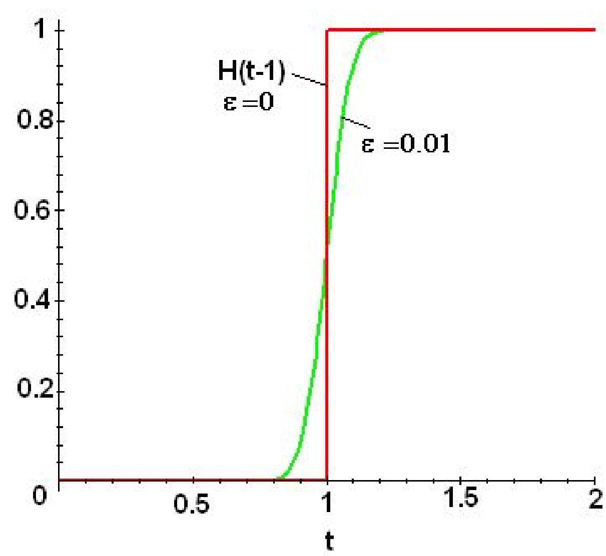
In several many areas of analysis one encounters discontinuous functions with your first exposure probably coming while studying Laplace transforms and their inverses. The best known of these functions are the Heaviside Step Function, the Dirac Delta Function, and the Staircase Function. Let us look at some of their properties. First start with the standard definitions-

$$H(t-a)=\begin{vmatrix}1\ if\ \ t>a\\0\ if\ \ t<a\end{vmatrix},\quad \delta(t-a)=\begin{vmatrix}\infty\ if\ \ t=a\\0\ if\ \ t\neq a\end{vmatrix},\quad and\quad S=\sum_{n=1}^{\infty}H(t-n)$$

To visualize these functions we can take the well known solution for heat conduction in a bar of infinite length as time approaches zero. This produces-

$$H(t-a)=\lim_{\varepsilon\to 0}\left\{\frac{1}{2}+\frac{1}{\sqrt{\pi}}\int_{u=0}^{\frac{(x-a)}{\sqrt{\varepsilon}}}\exp(-u^2)du\right\}$$

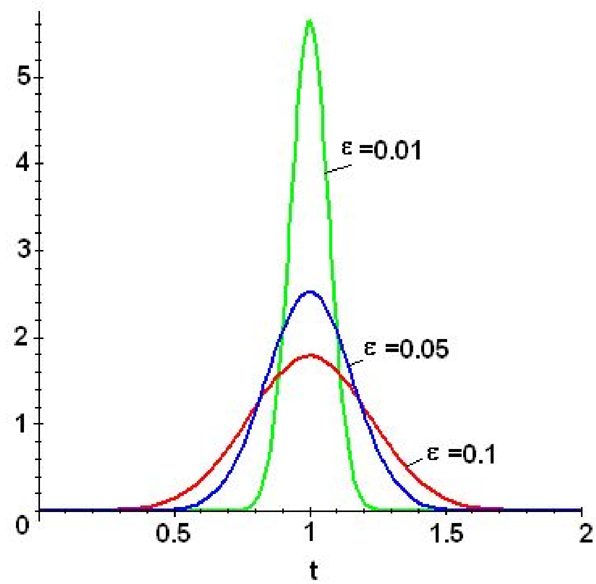
where the plot at a=1 for ε=0.01 and ε=0 looks as follows-



Clearly as ε goes to zero one sees a Heaviside Step Function with a discontinuity at t=1. When taking derivatives of the H(t-a) function it helps to take the derivative of the above function at small but finite ε and then let ε approach zero. Doing so we find the definition of the Dirac Delta Function as-

$$\delta(t-a) = \frac{dH(t-a)}{dt} = \lim_{\varepsilon \to 0} \left\{ \frac{\exp-[\frac{(t-a)^2}{\varepsilon}]}{\sqrt{\pi\varepsilon}} \right\}$$

Plotting the term within the curly bracket for different values of ε shows the Gaussian functions-



With the derivative of H(t-a) clearly approaching the spike form of the standard Dirac Delta Function as ε vanishes. Note that the area underneath these Gaussian curves is always unity, showing that the Dirac Delta Function clearly has a unit area lying underneath it. From this observation it also follows that-

$$f(a) = \int_{-\infty}^{\infty} f(t)\delta(t-a)dt$$

where f(t) is any continuous function of t.

To work out the derivative of the delta function we differentiate the above expression for finite ε and then take the limit. It yields-

$$\frac{d\delta(t-a)}{dt} = \lim_{\varepsilon \to 0} \left[\frac{-2(t-a)\exp-[(t-a)^2/\varepsilon]}{\sqrt{\pi\varepsilon^3}} \right]$$

and is a double hump distribution of odd symmetry about t=a. The area underneath the curve is zero. It is not always obvious what the integral of the product of a continuous function f(t) and the derivative of a delta function is. For example, one has-

$$\int_{-\infty}^{\infty} \sin(t)\frac{d\delta(t)}{dt}dt = -1 \quad but \quad \int_{-\infty}^{\infty} t^3 \frac{d\delta(t)}{dt}dt = 0$$

despite the fact that both functions f(t) are odd. To resolve this conundrum, one needs to integrate by parts. Doing so one has-

$$\int_{-\infty}^{\infty} f(t)\frac{d\delta(t-a)}{dt}dt = f(t)\delta(t-a)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(t-a)\frac{df(t)}{dt}dt = -\frac{df(a)}{dt}$$

from which the stated values of the two definite integrals given above follow.

The integral of the nth derivative of a Dirac Delta Function multiplied by a continuous function f(t) becomes-

$$\int_{-\infty}^{\infty} f(t)\frac{d^n\delta(t-a)}{dt^n}dt = (-1)^n \frac{d^n f(a)}{dt^n}$$

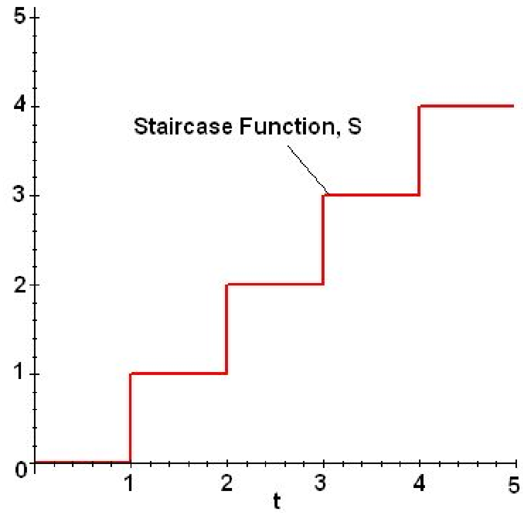
We thus have that-

$$\int_{0}^{1} t(t^2-1)\frac{d^2\delta(t-1/2)}{dt^2}dt = 3$$

Next, let us look at the staircase function which is constructed by stacking up of Heaviside Step Functions with each function moved one unit to the right. It reads-

$$S = H(t-1) + H(t-2) + H(t-3) + H(t-4) + \dots .$$

and its plot looks like this-



Note that the derivative of S are just a bunch of Dirac Delta Functions occurring at each positive integer t=n. You probably first encountered S during a course on Laplace transforms. Recall that-

$$Laplace(S) = \int_{t=0}^{\infty} [(H(t-1) + H(t-2) + H(t-3) + ..]\exp(-st)dt$$
$$= \frac{1}{s}\sum_{n=1}^{\infty} \exp(-ns) = \frac{1}{s[\exp(s)-1]}$$

where the sum is evaluated via the geometric series. The Laplace transform of the Heaviside step function is simply –

$$Laplace(H(t-a)) = \int_{t=a}^{\infty} \exp(-st)dt = \frac{\exp(-sa)}{s}$$

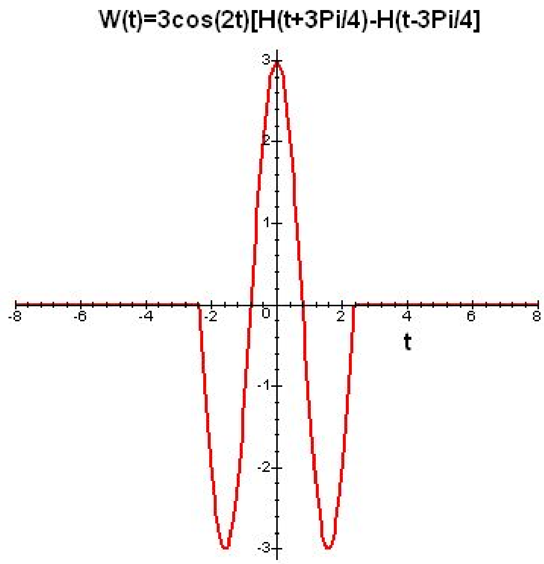
as can also be deduced from the Laplace transform for S. The Laplace transform of the Dirac Delta Function has perhaps the simplest form of all Laplace transforms, namely-

$$Laplace[(\delta(t-a)] = \int_{t=0}^{\infty} \delta(t-a)\exp(-st)dt = \exp(-sa)$$

Notice that one can construct various other discontinuous functions using δ(t-a), H(t-a), and S as a basis. Thus a window function, having a value of 3cos(2t) between t=-3π/4 and t=+3π/4 and a value of zero everywhere else, can be written as-

$$W(t) = 3\cos(2t)[H(t + 3\pi/4) - H(t - 3\pi/4)]$$

The plot of this function looks like this-



One can also construct an even symmetric Double Blip Rectangular Function where the pulses have height c and width (b-a). Mathematically one has-

$$DB = c\left[H(t+b) - H(t+a) + H(t+a) - H(t-b)\right]$$

Our MAPLE program for plotting this last function for a=2, b=3, and c=4 reads-

with(plots): with(inttrans): plot(4*(Heaviside(t+3)-Heaviside(t+2)+Heaviside(t-2)-Heaviside(t-3)), t=-5..5);

and produces the plot-