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Analytical Modeling of Sandwich Beams with Functionally Graded Core

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ABSTRACT: This study investigates several available sandwich beam theories for their suitability of application to one-dimensional sandwich plates with functionally graded core. Two equivalent single-layer theories based on assumed displacements, a higher-order theory, and the Fourier–Galerkin method are compared. The results are also compared with the finite element analysis. The core of the sandwich panel is functionally graded such that the density, and hence its stiffness, vary through the thickness. The variation of core Young's modulus is represented by a differentiable function in the thickness coordinate, but the Poisson's ratio is kept constant. A very good agreement is found among the Fourier–Galerkin method, the higher-order theory, and the finite element analysis.

KEY WORDS: functionally graded cores, functionally graded materials, sandwich panels.

INTRODUCTION

WITH THE DEVELOPMENTS in manufacturing methods [1–5] functionally graded materials (FGMs) seem to have great potential as core materials in sandwich structures. They possess properties that vary gradually with location within the material such as to optimize some function. New methodologies have to be developed to characterize FGMs, and to design and analyze structural components made with these materials. The methods should be such that they can be incorporated into available

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Figures 5–10 appear in color online: http://jsm.sagepub.com

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methods with minimal modifications. One of the important problems is that of response of structures made of FGMs to thermo-mechanical loads. Although FGMs are highly heterogeneous, it will be useful to idealize them as continua with properties changing smoothly with respect to the spatial coordinate. This will enable obtaining closed-form solutions to some fundamental solid mechanics problems, and also will help in developing finite element models of structures made of FGMs.

A considerable amount of literature exists on sandwich panels as they are used in large number of applications varying from high-performance composites in aerospace structures to low-cost materials for building constructions. The limitations of classical plate theory in describing complex problems (e.g., contact/impact problems, behavior of thick laminate plates) necessitated the development of higher-order theories. The term higher-order refers to the level of truncation of terms in a power series expansion of displacements about the thickness coordinate. The models investigated here can be classified into: single-layer theories (where the assumed displacement components represent the weighted-average through the thickness of sandwich panel), layerwise theories (where separate assumptions for displacements in each layer are made), and exact theories (equilibrium equations are solved without displacements assumptions). Although discrete-layer theories are more representative for sandwich construction than the single-layer theories, they experience computational difficulties from a large number of field variables in proportion to the number of layers.

Reissner [6] and Midlin [7] were the first to propose a plate theory that included the effect of shear deformation and that assumed linear longitudinal displacements and constant transverse displacements. Midlin [7] introduced the correction factor into the shear stress to account for the fact that the model predicts a uniform shear stress through the thickness of the plate. Yang et al. [8] extended Mindlin's theory for homogeneous plates to laminates consisting of arbitrary number of bonded layers. Based on the same model (Mindlin's theory), Whitney and Pagano [9] developed a theory for anisotropic laminated plates consisting of an arbitrary number of bonded anisotropic layers that includes shear deformation and rotary inertia. Displacement field is assumed to be linear in thickness coordinate. Since then, the plate theory was improved by including higher-order terms in displacements assumptions. Essenburg [10] assumed second-order transverse displacements and linear longitudinal displacements; Reissner [11] included third-order terms in the in-plane displacements' z-expansion; Lo et al. [12] included third-order in-plane and second-order out-of-plane terms. Reddy [13] developed a third-order shear deformation theory (TSDT) for composite laminates, based on assumed displacement fields (third-order in-plane and constant out-of-plane displacement).

Frostig et al. [14,15] developed a high-order theory to study the behavior of a sandwich beam with transversely flexible core based on variational principles. The main feature of the method is the higher-order displacement fields in the thickness coordinate. The longitudinal and the transverse deformations of the core are determined with the aid of constitutive equations of an isotropic material, and the compatibility conditions at the interfaces consist of non-linear expressions in thickness coordinate. Their formulation is based on the beam theory for face sheets and a two-dimensional elasticity theory for the core. The sandwich behavior is presented in terms of displacements in face sheets and shear stress of the core. Their model can be applied to sandwich structures with honeycomb or foam cores. The model was applied to the vibration analysis [16] and to sandwich structures with nonparallel skins [17]. Zhu and Sankar [18] derived an analytical model for a functionally graded (FG) beam with Young's modulus expressed as a polynomial in thickness coordinate using a combined Fourier series–Galerkin method.

In order to provide a simple tool for describing the behavior of a sandwich beam with FG core, this study compares four models found in the literature. These models are: first-order shear deformation theory (FSDT); TSDT, [13]; Fourier–Galerkin method developed by Zhu and Sankar [18]; and the higher-order theory of Frostig et al. [15]. A very good agreement is found between Zhu and Sankar model and Frostig model. The conclusions are supported with results from finite element analysis.

AN EQUIVALENT SINGLE-LAYER FIRST-ORDER SHEAR DEFORMATION THEORY

The simplest model investigated in the present study is an equivalent single-layer model or the FSDT [6,7] that includes a transverse shear deformation. The following kinematic assumptions are made:

$$\begin{aligned} u(x,z) &= u_0(x) + z\psi(x) \\ w(x,z) &= w_0(x) \end{aligned} \tag{1}$$

where u is the displacement in the horizontal direction, x, and w is the displacement in the vertical direction, z. u_0, w_0, ψ are unknown functions to be determined using the equilibrium equations of the first-order theory. The dimensions of the sandwich beam are shown in Figure 1. The length of the beam is L, the core thickness is h, the top face sheet thickness is h_t and the bottom face sheet thickness is h_b.

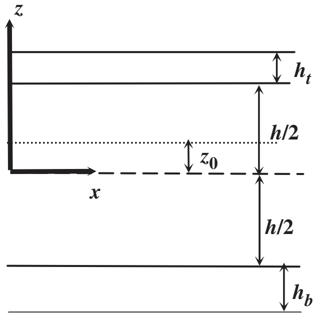


Figure 1. *The sandwich beam geometry.*

If the core material is isotropic at every point and the principal material directions coincide with the x- and z-axes, the plane strain constitutive relations are:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} c_{11}(z) & c_{13}(z) & 0 \\ c_{13}(z) & c_{33}(z) & 0 \\ 0 & 0 & c_{55}(z) \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{zz} \\ \gamma_{xz} \end{Bmatrix}$$

$$= \frac{E(z)}{(1+\nu)(1-2\nu)} \begin{pmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{pmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{zz} \\ \gamma_{xz} \end{Bmatrix} \quad (2)$$

or

$$\boldsymbol{\sigma}(x,z) = \mathbf{c}(z)\ \boldsymbol{\varepsilon}(x,z).$$

The variation of Young's modulus E in the thickness direction is assumed to be a polynomial in z. e.g.,

$$E(z) = E_0\left(a_1\left(\frac{z}{h}\right)^4 + a_2\left(\frac{z}{h}\right)^3 + a_3\left(\frac{z}{h}\right)^2 + a_4\left(\frac{z}{h}\right) + 1\right). \quad (3)$$

where E_0 is the Young's modulus at $z=0$ and a_1, a_2, a_3, and a_4 are material constants.

In order to calculate the flexural rigidity of the cross-section, the position of the neutral axis z_0 must be found. It is given by the coordinate system for which the bending-stretching coupling term vanishes:

$$B_{11} = \int_{-(h_b+h/2+z_0)}^{h_t+h/2-z_0} z c_{11}(z) dz = 0 \Rightarrow z_0. \tag{4}$$

Constitutive equations of the first-order theory take the following form:

$$\begin{cases} N_x = A_{11} \dfrac{du_0}{dx} \\ M_x = D_{11} \dfrac{d\psi}{dx} \\ V_z = S\left[\psi + \dfrac{dw_0}{dx}\right] \end{cases} \tag{5}$$

where A_{11} is the extensional stiffness, D_{11} is the bending stiffness, and S is the transverse shear stiffness given by

$$\begin{aligned} A_{11} &= \int_{-(h_b+h/2+z_0)}^{(h_b+h/2+z_0)} c_{11}(z) dz \\ D_{11} &= \int_{-(h_b+h/2+z_0)}^{h_t+h/2-z_0} z^2 c_{11}(z) dz. \\ S &= \int_{-(h/2+z_0)}^{h/2-z_0} c_{55}(z) dz \end{aligned} \tag{6}$$

For a given set of external loads and boundary conditions, axial force resultant N_x, bending moment resultant M_x, and shear resultant V_z can be calculated. Then using system (5) the deformations are obtained. The deformations can be integrated to obtain the displacements and rotation.

This model is applied for a sandwich beam with FG core. The main feature of the model is that the transverse shear strain is constant through the thickness of the beam. Results and discussions of this model are presented in the last section.

AN EQUIVALENT SINGLE-LAYER THIRD-ORDER SHEAR DEFORMATION THEORY

Reddy [13] developed a TSDT for composite laminates, based on assumed displacement fields and using the principle of virtual displacements.

Reddy [19] reviewed several other third-order theories and showed that they are special cases of the theory proposed in his study. Reddy [20] and Reddy and Cheng [21] expanded TSDT for the analysis of FG plates.

Here, a third-order equivalent single-layer model based on Reddy's assumption of vanishing of transverse shear stresses on the bounding planes is investigated. The displacement field is assumed to be:

$$\begin{aligned} u(x,z) &= u_0(x) + z\psi(x) + z^2\chi(x) + z^3\phi(x) \\ w(x,z) &= w_0(x) \end{aligned} \tag{7}$$

where u_0 and w_0 are displacements along middle axis, $z=0$. Functions χ and Φ are eliminated using the assumption of zero shear stresses at top and bottom:

$$\begin{aligned} \chi(x) &= \frac{h_\mathrm{t} - h_\mathrm{b}}{h^*}\left[\psi(x) + \frac{dw_0}{dx}\right] \\ \phi(x) &= -\frac{2}{3h^*}\left[\psi(x) + \frac{dw_0}{dx}\right] \end{aligned} \tag{8}$$

where

$$h^* = 2h_\mathrm{t}h_\mathrm{b} + h_\mathrm{t}h + h_\mathrm{b}h + \frac{h^2}{2}. \tag{9}$$

If the top face sheet thickness h_t equals the bottom face sheet thickness h_b, then $\chi(x)=0$.

Axial force resultant, bending moment resultant, and shear resultant are calculated as follows:

$$\begin{cases} N_x = b\displaystyle\int_{-(h/2+h_\mathrm{b})}^{h/2+h_\mathrm{t}} \sigma_{xx}(x,z)dz = \varepsilon_0(x)A_N + \frac{d\psi}{dx}B_N + \frac{d^2w}{dx^2}C_N \\ M_x = b\displaystyle\int_{-(h/2+h_\mathrm{b})}^{h/2+h_\mathrm{t}} z\sigma_{xx}(x,z)dz = \varepsilon_0(x)A_M + \frac{d\psi}{dx}B_M + \frac{d^2w}{dx^2}C_M \\ V_z = b\displaystyle\int_{-(h/2+h_\mathrm{b})}^{h/2+h_\mathrm{t}} \tau_{xz}(x,z)dz = \psi(x)B_V + \frac{dw}{dx}C_V \end{cases} \tag{10}$$

where b is the beam width, $\varepsilon_0 = du_0/dx$ and stiffness coefficients A_N through C_V are given by:

$$A_N = b\int_{-((h/2)+h_b)}^{(h/2)+h_t} c_{11}(z)dz \quad B_N = b\int_{-((h/2)+h_b)}^{(h/2)+h_t} c_{11}(z)\left[z + z^2\frac{h_t - h_b}{h^*} - z^3\frac{2}{3h^*}\right]dz$$

$$C_N = b\int_{-((h/2)+h_b)}^{(h/2)+h_t} c_{11}(z)\left[z^2\frac{h_t - h_b}{h^*} - z^3\frac{2}{3h^*}\right]dz$$

$$A_M = b\int_{-((h/2)+h_b)}^{(h/2)+h_t} zc_{11}(z)dz \quad B_M = b\int_{-((h/2)+h_b)}^{(h/2)+h_t} zc_{11}(z)\left[z + z^2\frac{h_t - h_b}{h^*} - z^3\frac{2}{3h^*}\right]dz$$

$$C_M = b\int_{-((h/2)+h_b)}^{(h/2)+h_t} zc_{11}(z)\left[z^2\frac{h_t - h_b}{h^*} - z^3\frac{2}{3h^*}\right]dz$$

$$B_V = b\int_{-((h/2)+h_b)}^{(h/2)+h_t} c_{55}(z)\left[1 + 2z\frac{h_t - h_b}{h^*} - z^2\frac{2}{h^*}\right]dz$$

$$C_V = b\int_{-((h/2)+h_b)}^{(h/2)+h_t} c_{55}(z)\left[2z\frac{h_t - h_b}{h^*} - z^2\frac{2}{h^*}\right]dz. \tag{11}$$

The following set of constitutive equations is obtained:

$$\begin{bmatrix} A_N & B_N & C_N \\ A_M & B_M & C_M \\ 0 & B_V & C_V \end{bmatrix} \begin{Bmatrix} \dfrac{du_0(x)}{dx} \\ \dfrac{d\psi(x)}{dx} \\ \dfrac{dw(x)}{dx} \end{Bmatrix} = \begin{Bmatrix} N_x(x) \\ M_x(x) \\ V_z(x) \end{Bmatrix}. \tag{12}$$

For a given set of external loads and boundary conditions axial force resultant N_x, bending moment resultant M_x, and shear resultant V_z can be calculated. Then using the system of equations (12) the displacements are obtained.

This model is applied for a sandwich beam with FG core. The main drawback of the model is given by the fact that the transverse shear strain has a quadratic variation with respect to the thickness coordinate, whereas the actual strains in a graded core may be different depending on the gradation. Results from this model are compared with others in the last section.

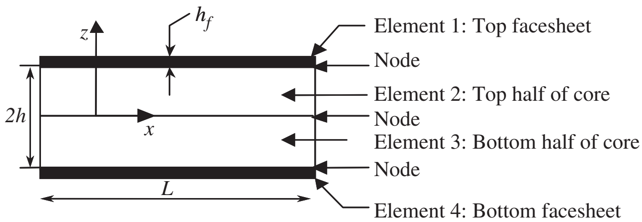


Figure 2. *Sandwich beam with functionally graded core divided into four elements.*

FOURIER–GALERKIN METHOD

Zhu and Sankar (2004) derived an analytical model for a FG beam using a combined Fourier series–Galerkin method. They considered problems in which the elastic constants were expressed as polynomials in the thickness coordinate, z. It should be mentioned that the method solves the elasticity equations, and hence no assumptions are made regarding the displacement field. In the present work, the model is applied to a sandwich beam with FG core.

The dimensions of the sandwich beam are shown in Figure 2. The length of the beam is L, the core thickness is h, and the face sheet thicknesses are h_{f}. The beam is divided into four parts or elements: the top face sheet, top half of the core, bottom half of the core, and the bottom face sheet.

The coordinate systems for each element are chosen at the interface (Figure 3), because it will be convenient to enforce displacement compatibility and continuity of tractions between elements at the interface nodes. The face sheets are assumed to be homogeneous and isotropic. The core is orthotropic at every point. The elasticity equations are formulated separately for each element, and compatibility of displacements and continuity of tractions are enforced at each interface (node) to obtain the displacement and stress field in the sandwich beam. This procedure is analogous to assembling element stiffness matrices to obtain global stiffness matrix in finite element analysis.

It is assumed that the top face sheet is subjected to normal tractions such that:

$$\sigma_{zz}(x,0) = p_{\mathrm{a}} \sin(\xi x) \tag{13}$$

where

$$\xi = \frac{n\pi}{L}, \quad n = 1, 3, 5, \ldots \tag{14}$$

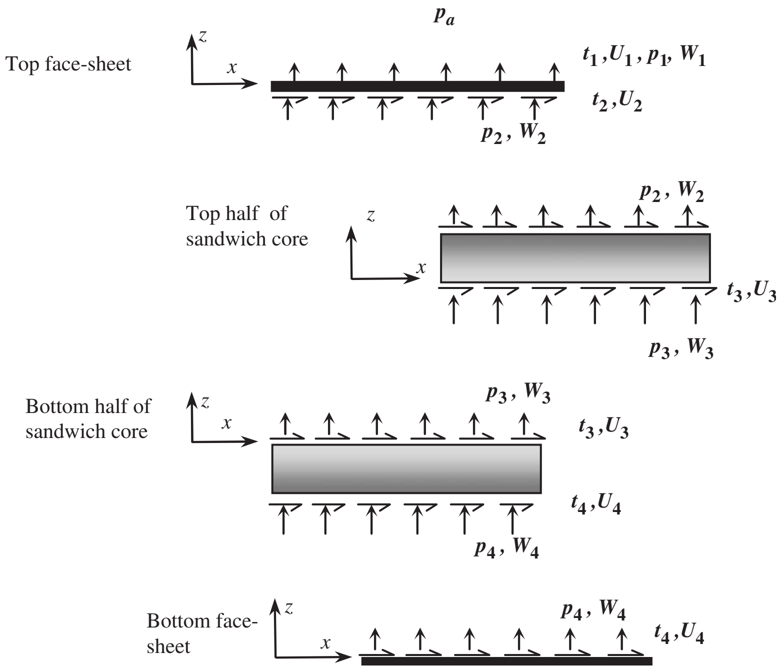


Figure 3. *Traction forces and displacements at the interfaces of each element in the FGM sandwich beam.*

and p_{a} is known. Since n is assumed to be odd, the loading is symmetric about the center of the beam. The loading given by Equation (13) is of practical significance because any arbitrary loading can be expressed as a Fourier series involving terms of the same type. Boundary conditions of the beam at $x=0$ and $x=L$ are $w(0,z)=w(L,z)=0$ and σ_{xx} $(0, z)=\sigma_{xx}$ $(L, z)=0$, which corresponds to simple support conditions in the context of beam theory.

In this study, a brief description of the procedures is presented in order to obtain the stiffness matrix of top half of the core. The derivation of the stiffness matrices of other elements follows the same procedures.

The differential equations of equilibrium for the top half of the core are:

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = 0 \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = 0. \end{aligned} \tag{15}$$

The constitutive relation is given by Equation (2) and the Young's modulus variation is given by Equation (3). It is assumed that width in the y-direction is large and plain strain assumption can be used.

The displacements can be expressed as:

$$\begin{aligned} u(x,z) &= U(z)\cos\xi x \\ w(x,z) &= W(z)\sin\xi x. \end{aligned} \tag{16}$$

In order to solve Equations (15), the Galerkin method is used. The solutions are assumed in the form of polynomials in z as follows:

$$\begin{aligned} U(z) &= c_1\phi_1(z) + c_2\phi_2(z) + c_3\phi_3(z) + c_4\phi_4(z) + c_5\phi_5(z) \\ W(z) &= b_1\phi_1(z) + b_2\phi_2(z) + b_3\phi_3(z) + b_4\phi_4(z) + b_5\phi_5(z) \end{aligned} \tag{17}$$

where ϕs are basis functions, and bs and cs are coefficients to be determined. For simplicity, basis functions are chosen as:

$$\phi_1(z) = 1 \quad \phi_2(z) = z \quad \phi_3(z) = z^2 \quad \phi_4(z) = z^3 \quad \phi_5(z) = z^4. \tag{18}$$

Substituting the approximate solution (16) in the governing differential equations (15) and minimizing the residuals (by equating their weighted averages to zero) the stiffness matrix of the top half of the FG core that relates the surface tractions to the surface displacements is determined:

$$\begin{pmatrix} T_2 \\ S_2 \\ T_3 \\ S_3 \end{pmatrix} = \left[S^{(2)}\right] \begin{pmatrix} U_2 \\ W_2 \\ U_3 \\ W_3 \end{pmatrix}. \tag{19}$$

In order to satisfy equilibrium, the contributions of the different tractions at each interface should sum to zero. Enforcing the balance of force and compatibility of displacements at the interfaces allows one to assemble the stiffness matrices of the four elements to obtain a global stiffness matrix S:

$$\begin{aligned} [\mathbf{S}]\lfloor U_1 \quad W_1 \quad U_2 \quad W_2 \quad U_3 \quad W_3 \quad U_4 \quad W_4 \quad U_5 \quad W_5 \rfloor^{\mathrm{T}} \\ = \lfloor 0 \quad p_a \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \rfloor^{\mathrm{T}}. \end{aligned} \tag{20}$$

The displacements at the interfaces U_1, $W_1 \ldots W_5$, are obtained by solving Equation (20). The displacement fields along with the constitutive relations are used to obtain the stress field in each element.

A HIGHER-ORDER SHEAR DEFORMATION THEORY

Frostig et al. [15] developed a higher-order theory for a sandwich beam with a transversely flexible core that uses a beam theory for the face sheets and two-dimensional elasticity equations for the core. Swanson [22] addressed details of implementation of the Frostig model and presented solutions for several cases. The main feature of the method is the higher-order displacement fields in the thickness coordinate: second order for the transverse displacements and third order for the longitudinal displacements. Another advantage of this model, as a model based on variational principle, is that the boundary conditions are obtained uniquely as a part of the derivation. The main difference between this model and the previous one is that the higher-order displacements are derived and not assumed. The equations developed by Frostig et al. [15] and modified for a FG core are briefly presented here.

The dimensions of the sandwich beam and the coordinate systems are shown in Figure 4. Constitutive relations (which assume isotropic elastic behavior) for the face sheets and for the core are given by:

$$\begin{aligned} \text{Top face sheet: } N^{\mathrm{t}}_{xx} &= A^{\mathrm{t}}_{11} u_{0\mathrm{t},x} \quad M^{\mathrm{t}}_{xx} = -D^{\mathrm{t}}_{11} w_{\mathrm{t},xx} \\ \text{Bottom face sheet: } N^{\mathrm{b}}_{xx} &= A^{\mathrm{b}}_{11} u_{0\mathrm{b},x}, \quad M^{\mathrm{b}}_{xx} = -D^{\mathrm{b}}_{11} w_{\mathrm{b},xx} \\ \text{Core: } \tau(x,z) &= G_{\mathrm{c}}(z)\gamma(x,z) \\ \sigma_{zz}(x,z) &= \frac{(1-\nu)E_{\mathrm{c}}(z)}{(1+\nu)(1-2\nu)} w_{\mathrm{c},z}(x,z) \end{aligned} \tag{21}$$

where N^i_{xx} are the resultant axial forces in the face sheets, M^i_{xx} are the bending moments in the face sheets, A^i_{11} and D^i_{11} are, respectively, the axial

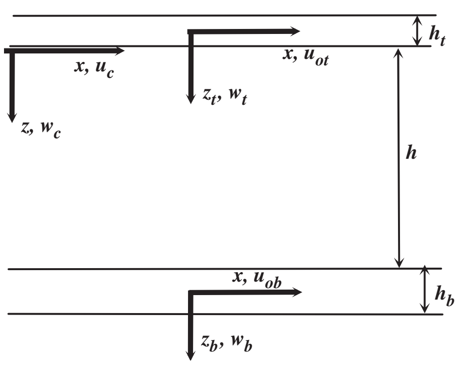


Figure 4. *Sandwich beam geometry. Length of the beam is L, the core thickness is h, the top face sheet thickness is h_t and the bottom face sheet thickness is h_b.*

and flexural rigidities for the face sheets, u_{0i}, w_i are face sheets longitudinal and vertical displacements at the centroid ($i=\mathrm{t}$ for top face sheet and $i=\mathrm{b}$ for bottom face sheet):

$$A_{11}^i = \int_{-h_i/2}^{h_i/2} c_{11}^i(z)dz$$
$$D_{11}^i = \int_{-h_i/2}^{h_i/2} z^2 c_{11}^i(z)dz. \tag{22}$$

Linear variation for Young's modulus and shear modulus are assumed:

$$E(z) = az + b$$
$$G(z) = a_1 z + b_1. \tag{23}$$

Governing differential equations, boundary conditions, and continuity conditions are derived based on variational principles. The equilibrium equations for the core are:

$$\sigma_{x,x} + \tau_{xz,z} = 0$$
$$\tau_{xz,x} + \sigma_{z,z} = 0. \tag{24}$$

The following assumptions and conditions are used to derive the core displacement field: (a) compatibility of the displacements at the core–face sheet interfaces; (b) the core is transversely flexible, i.e., it has much lower stiffness relative to the face sheets, so that the core does not carry any longitudinal stresses; and (c) the shear stress is nearly constant through the core. Then the core displacement fields are derived as follows:

$$u_{\mathrm{c}}(x,z) = \tau(x)\frac{1}{a_1}\ln\left(\frac{a_1}{b_1}z+1\right) + \tau_{,xx}(x)\frac{1}{2a}z^2 - w_{\mathrm{t},x}(x)\left(z+\frac{h_{\mathrm{t}}}{2}\right) + u_{o\mathrm{t}}$$
$$+\left(w_{\mathrm{b},x} - w_{\mathrm{t},x} + \frac{h}{a}\tau_{,xx}(x)\right)\frac{(z+(b/a))\ln((a/b)z+1) - z}{\ln((a/b)h+1)} \tag{25}$$
$$w_{\mathrm{c}}(x,z) = -\frac{z}{a}\tau_{,x}(x) + \frac{\ln((a/b)z+1)}{\ln((a/b)h+1)}\left(w_{\mathrm{b},x} - w_{\mathrm{t},x} + \frac{h}{a}\tau_{,xx}(x)\right) + w_{\mathrm{t}}$$

where u_{0i} and w_i are longitudinal and vertical displacements of the centroid of each face sheet ($i=\mathrm{t}$ for top face sheet and $i=\mathrm{b}$ for bottom face sheet); ν is core Poisson's ratio. In the above expression a linear core Young's modulus was assumed: $E_{\mathrm{c}}(z)=az+b$. Similar relations can be obtained for any Young's modulus variation expressed by a differentiable function.

Governing differential equations, boundary conditions and continuity conditions are derived based on the variational principle. Governing equations, written in terms of transverse displacements $w_{\rm t}$ and $w_{\rm b}$ of the top and bottom face sheets and shear stress of the core, τ are:

$$\begin{cases} \alpha_1 w_{{\rm t},xxxx}(x) + \beta_1 \tau_{,x}(x) + \eta_1 w_{\rm b}(x) + \mu_1 w_{\rm t}(x) = q_{\rm t}(x) - m_{{\rm t},x}(x) \\ \alpha_2 w_{{\rm b},xxxx}(x) + \beta_2 \tau_{,x}(x) + \eta_2 w_{\rm b}(x) + \mu_2 w_{\rm t}(x) = q_{\rm b}(x) - m_{{\rm b},x}(x) \\ \alpha_3 w_{{\rm t},xxx}(x) + \beta_3 w_{{\rm b},xxx}(x) + \eta_3 \tau_{,xxxx}(x) + \mu_3 \tau_{,xx}(x) + \omega_3 \tau(x) \\ \quad = \dfrac{1}{A_{11}^{\rm t}} n_{\rm t}(x) - \dfrac{1}{A_{11}^{\rm b}} n_{\rm b}(x) \end{cases} \tag{26}$$

where q_i, m_i, and n_i are distributed pressures, moments, and axial forces applied on top ($i = {\rm t}$) and bottom ($i = {\rm b}$) face sheets and the coefficients are given by:

$$\begin{aligned}
&\alpha_1 = D_{11}^{\rm t} \quad \beta_1 = -d\left[\frac{h_{\rm t}}{2} + \frac{h}{\ln((a/b)h+1)} - \frac{b}{a}\right] \quad \eta_1 = -\frac{ad}{\ln((a/b)h+1)} \\
&\mu_1 = \frac{ad}{\ln((a/b)h+1)} \quad \alpha_2 = D_{11}^{\rm b} \quad \beta_2 = d\left[-h - \frac{h_{\rm b}}{2} + \frac{h}{\ln((a/b)h+1)} - \frac{b}{a}\right] \\
&\eta_2 = \frac{ad}{\ln((a/b)h+1)} \quad \mu_2 = -\frac{ad}{\ln((a/b)h+1)} \\
&\alpha_3 = -h - \frac{h_{\rm t}}{2} + \frac{(h+(b/a))\ln((a/b)h+1) - h}{\ln((a/b)h+1)} \\
&\beta_3 = -\frac{h_{\rm b}}{2} - \frac{(h+(b/a))\ln((a/b)h+1) - h}{\ln((a/b)h+1)} \\
&\eta_3 = \frac{1}{2a}h^2 - \frac{h}{a}\frac{(h+(b/a))\ln((a/b)h+1) - h}{\ln((a/b)h+1)} \quad \mu_3 = \frac{1}{a_1}\ln\left(\frac{a_1}{b_1}h + 1\right) \\
&\omega_3 = -d\left(\frac{1}{A_{11}^{\rm t}} + \frac{1}{A_{11}^{\rm b}}\right)
\end{aligned} \tag{27}$$

where d is the beam width.

In order to obtain the homogeneous solution for (26), the following characteristic equation is derived:

$$\begin{vmatrix} \alpha_1\lambda^4 + \mu_1 & \eta_1 & \beta_1\lambda \\ \mu_2 & \alpha_2\lambda^4 + \eta_2 & \beta_2\lambda \\ \alpha_3\lambda^3 & \beta_3\lambda^3 & \eta_3\lambda^4 + \mu_3\lambda^2 + \omega_3 \end{vmatrix} = 0. \tag{28}$$

Denoting with λ_j, $j = 1, \ldots, 12$, the roots of characteristic equation, the following rescaling of constants needs to be done in order to avoid numerical difficulties (given by the case when $\text{Real}(\lambda_j) > 0 \Rightarrow \exp(\lambda_j x) \underset{x \to L}{\longrightarrow} \infty$):

$$\text{(i) If } \text{Re}(\lambda_j) < 0 \quad \Rightarrow \quad \text{solution} = c_j e^{\text{Re}(\lambda_j)x}$$

$$\text{(ii) If} \text{Re}(\lambda_j) > 0 \quad \Rightarrow \quad \text{solution} = c_j \frac{e^{-\lambda_j L}}{e^{-\lambda_j L}} e^{\text{Re}(\lambda_j)x} = \tilde{c}_j e^{[-\lambda_j(L-x)]}. \tag{29}$$

Forstig's theory needs the above described rescaling of constants in order to avoid numerical difficulties. For a given set of external loads and boundary conditions the governing system of differential equations (26) can be solved. Results and discussions are presented in the next section.

RESULTS AND DISCUSSION

A simply supported sandwich beam of length $L = 0.3$ m, core thickness $h = 20 \times 10^{-3}$ m, and face sheets thickness $h_f = 0.3 \times 10^{-3}$ m is considered to investigate the effects of varying core properties through the thickness. The face sheet Young's modulus was chosen as 50 GPa. The face sheets Poisson's ratio is $\nu_f = 0.25$ and the core Poisson's ratio is $\nu = 0.35$.

Two cases are considered:

(a) A simply supported sandwich structure with FG symmetric core about the midplane under a uniform distributed load, $p = 1\ \text{N/m}^2$
(b) A simply supported sandwich structure with FG asymmetric core about the midplane under a sinusoidal distributed load given by $p(x) = \sin(\pi x/L)\ \text{N/m}^2$.

Figure 5 depicts the variation of core elastic moduli through the thickness of the core. For case (a), the core Young's modulus E has a linear symmetric variation with respect to thickness coordinate, z. $E = 50$ MPa at the middle core and $E = 500$ MPa at the core–top face sheet interface (as well as at the core–bottom face sheet interface). For case (b) the core Young's modulus E has a linear asymmetric variation with respect to thickness coordinate. $E = 50$ MPa at the core–bottom face sheet interface and $E = 500$ MPa at the core–top face sheet interface.

Using the commercial finite element software ABAQUS$^{\text{TM}}$ [23] a 2D finite element model was created to model problem (a). The FG core was partitioned through the thickness into 20 strips with constant properties. Four elements were considered through the thickness of each strip and two elements were considered through the thickness of the face sheets.

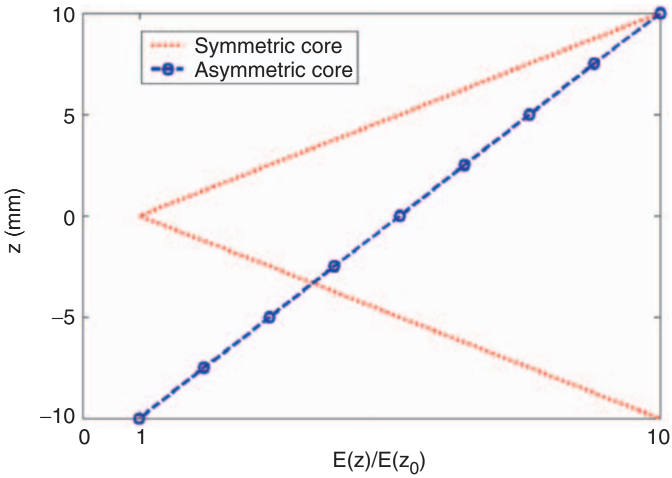


Figure 5. *Non-dimensional core modulus: symmetric core about the centerline and asymmetric core about the centerline.*

The elements considered were 2D, quadratic, plane strain elements. Boundary conditions assume $w(0,z) = w(L,z) = 0$.

Figures 6–10 present a comparison for five models: the equivalent single-layer first-order and third-order shear deformation theories, Fourier series–Galerkin method (Sankar model), Frostig model, and a finite element model. For problem (a), which is the symmetric core under a uniform distributed load, a very good agreement between Sankar model and FE model was found. As the core is symmetric about the mid-plane, the variation of displacements, strains, and stresses are symmetric with respect to the thickness coordinate.

Figure 6 presents deflections at bottom face sheet–core interface. The FSDT and the TSDT beams are stiffer than that predicted by the Sankar model.

Comparison of longitudinal displacements in the core at the same cross-section ($L/4$) for the two cases is presented in Figure 7(a) and (b). A perfect agreement was found between the Sankar model and the finite element model for Problem (a), and between Sankar model and the Frostig model for Problem (b). For both problems, as expected, FSDT gives a linear variation of displacements.

Bending stresses (Figure 8(a) and (b)) in the core at a given cross-section ($x = L/4$) are almost the same for all models except TSDT. TSDT overpredicts the bending stress values at core–face sheets interfaces for the

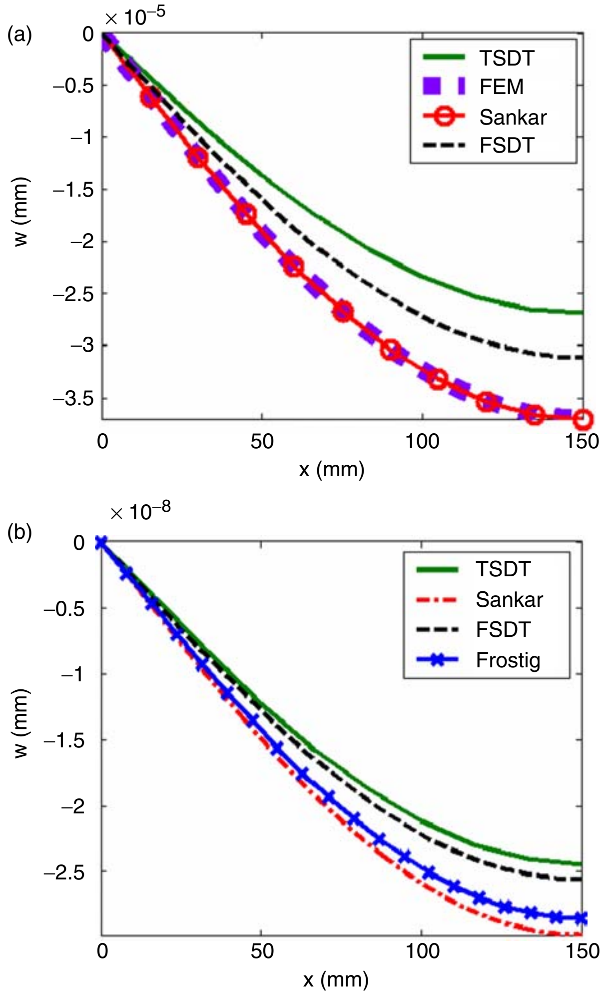
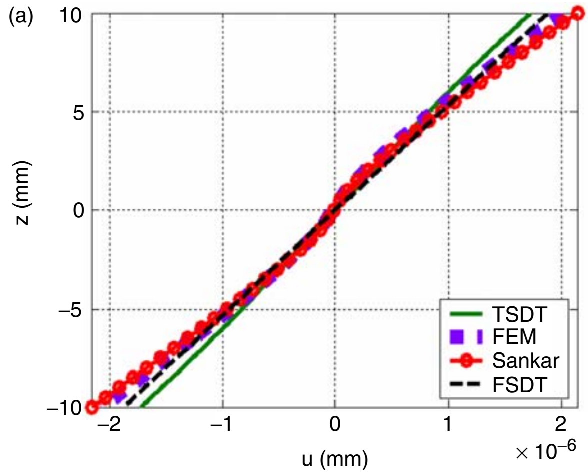


Figure 6. *Comparison of deflections: (a) symmetric core about the centerline under uniform distributed load $p = 1$ N/m^2 and (b) asymmetric core about the centerline under a distributed load given by $p(x) = \sin(\pi x/L)$ N/m^2.*

symmetric core case. For the asymmetric core, a larger compression value is found at the top core where the load is applied and the Young's modulus is larger.

Comparison of shear strain in the core at a given cross-section ($L/4$) is presented in Figure 9(a) and (b). Sankar model, Frostig model, and the finite element solutions present a $1/z$ type variation for the core shear strain (this is



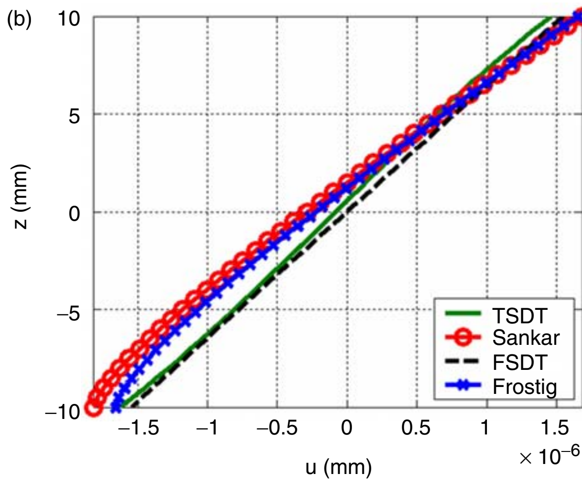


Figure 7. *Comparison of longitudinal displacement in the core at $x = L/4$: (a) symmetric core about the centerline under uniform distributed load $p = 1\ N/m^2$ and (b) asymmetric core about the centerline under a distributed load given by $p(x) = \sin(\pi x/L)\ N/m^2$.*

correct because the shear stress is constant though the core thickness and elastic modulus is linear with respect to the core thickness). The FSDT gives a constant shear strain while the TSDT gives a quadratic shear strain with respect to the thickness coordinate.

The same conclusion was reached for the shear stresses in the core at the cross-section given by $x = L/4$ (Figure 10). Sankar model, Frostig model,

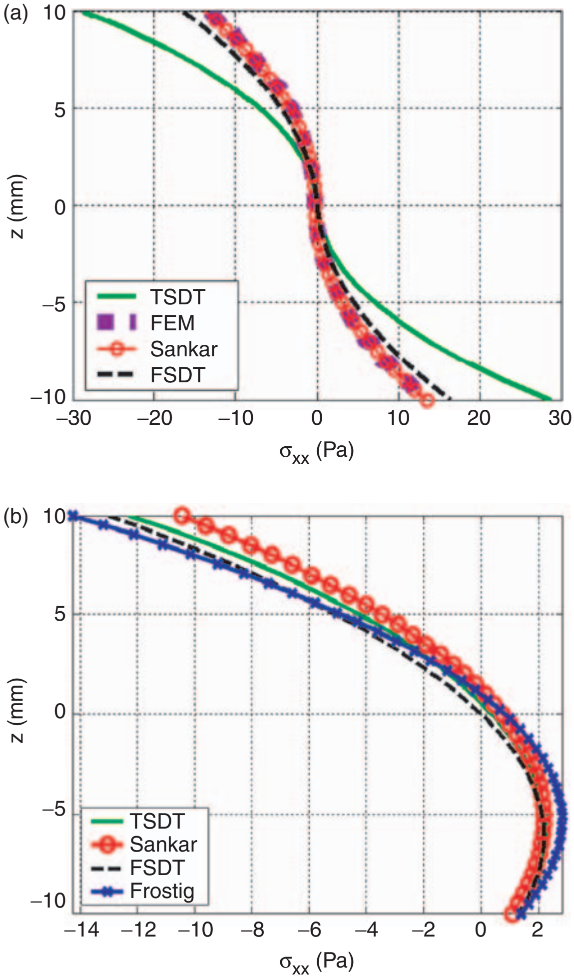
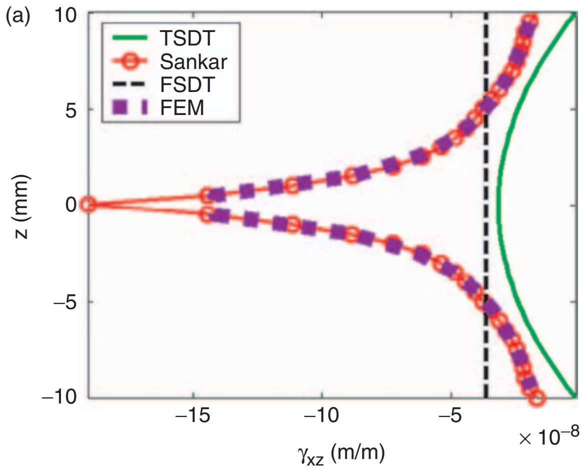


Figure 8. *Comparison of longitudinal stress in the core at $x = L/4$: (a) symmetric core about the centerline under uniform distributed load $p = 1\ N/m^2$ and (b) asymmetric core about the centerline under a distributed load given by $p(x) = \sin(\pi x/L)\ N/m^2$.*

and the finite element solution present an almost constant shear stress, whereas the FSDT gives a linear shear stress and the TSDT gives a cubic variation of shear stresses with respect to the thickness coordinate. The equivalent single-layer theories are not accurate for shear stresses because they are not based on two-dimensional equilibrium equations (24). In order to improve the accuracy for shear (strains and stresses) in single-layer



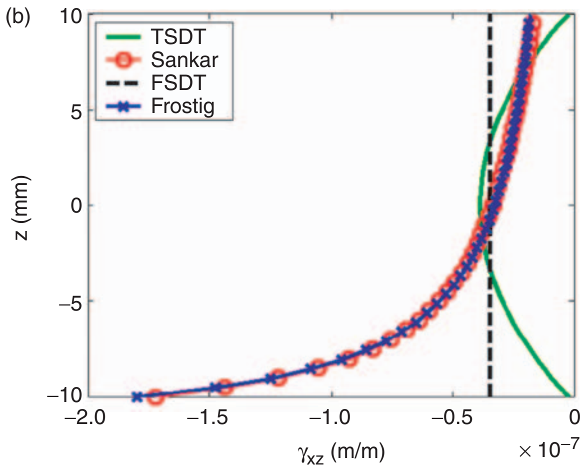


Figure 9. *Comparison of shear strain in the core at $x = L/4$: (a) symmetric core about the centerline under uniform distributed load $p = 1\,\mathrm{N/m^2}$ and (b) asymmetric core about the centerline under a distributed load given by $p(x) = \sin(\pi x/L)\ \mathrm{N/m^2}$.*

theories (both first order and third order) the shear stress was obtained by integrating the equilibrium equations (4) for the core and the bending stress previously derived (Figure 8):

$$\sigma_{xx,x} + \tau_{xz,z} = 0 \ \Rightarrow \quad \tau_{xz}(x,z) = -\int_0^z \sigma_{xx,x}(x,\xi)d\xi + \tau_{xz}(x,0). \tag{30}$$

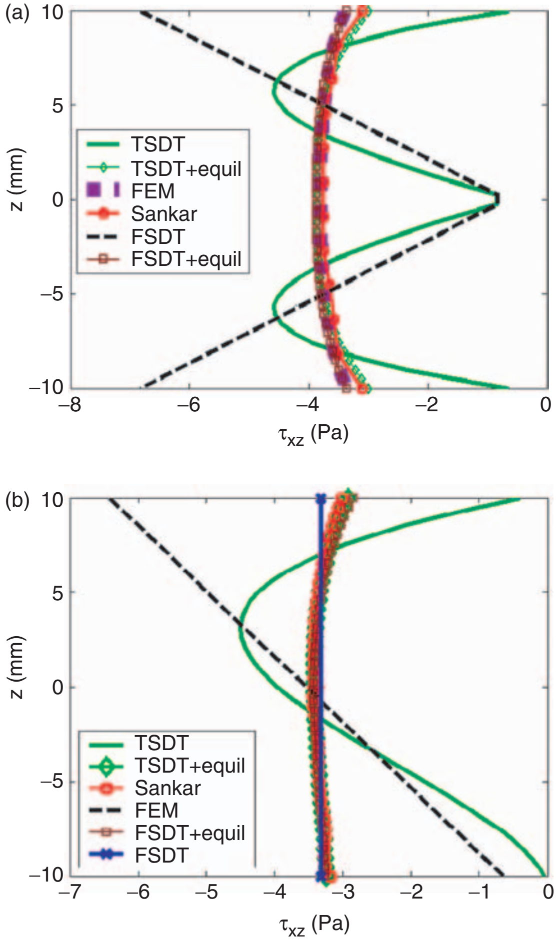


Figure 10. *Comparison of shear stress in the core at* $x = L/4$*: (a) symmetric core about the centerline under uniform distributed load* $p = 1\ N/m^2$ *and (b) asymmetric core about the centerline under a distributed load given by* $p(x) = \sin(\pi x/L)\ N/m^2$.

Figure 10 includes both shear stresses: obtained from single-layer theories and that obtained from equilibrium equation (30). The results are identical with those obtained based on Sankar model and the finite element model.

CONCLUSIONS

In order to describe the behavior of a sandwich structure with FG core under a distributed load, several methods can be used. This study compares four analytical models and a finite element solution. The simplest model, a combination between FSDT and equilibrium equations for the core (used in order to obtain accurate shear stresses) gives relatively good results. This method yields linearly varying longitudinal displacements and slightly smaller deflections. The TSDT overpredicts the bending stresses at the core–face sheets interfaces, and needs to be used in combination with equilibrium equations in order to predict constant shear stress for the FG core. Higher-order theories, e.g., Frostig et al. [15] and Zhu and Sankar (2004), require more computational effort, but give the most accurate results.

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