Computers & Structures Vol. 20, No. 6, pp. 1009–1012, 1985
Printed in Great Britain.

0045–7949/85 \$3.00 + .00
Pergamon Press Ltd.

AN EFFICIENT NUMERICAL ALGORITHM FOR TRANSVERSE IMPACT PROBLEMS

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(*Received* 9 *January* 1984; *received for publication* 23 *March* 1984)

Abstract—Transverse impact problems in which the elastic and plastic indentation effects are considered, involve a nonlinear integral equation for the contact force, which, in practice, is usually solved by an iterative scheme with small increments in time. In this paper, a numerical method is proposed wherein the iterations of the nonlinear problem are separated from the structural response computations. This makes the numerical procedures much simpler and also efficient. The proposed method is applied to some impact problems for which solutions are available, and they are found to be in good agreement. The effect of the magnitude of time increment on the results is also discussed.

1. INTRODUCTION

The earliest work on transverse impact of a beam which also took the local indentation into account was done by Timoshenko in 1913. This work has been cited by several authors (e.g. [1, 2]). The primary objective in such impact problems is the computation of the impact force history. By defining the local indentation as the difference in displacement of the impactor and that of the beam at the impact point, one can write down the equations of motion of the impactor and the beam. An integral equation for the contact force is obtained by assuming a contact law that relates the contact force and the amount of indentation. The Hertzian type of contact law, $F = k\alpha^{1.5}$, is assumed to hold good for isotropic materials. In the above equation F is the contact force, α is the indentation, and k is the contact coefficient. Because of the above relation the integral equation for the contact force becomes nonlinear, and has to be solved by an iterative scheme with small increments in time. Once the impact force history is known, the beam response can easily be computed. This method is straightforward, but too lengthy and tedious. It can be applied to any structure, e.g. Karas[3] applied this method for the problem of plate impact.

An interest in the low velocity impact problems has been revived with the advent of composite materials. For example, the impact resistance of graphite epoxy composites, which find considerable application in aircraft structures, is much lower than that of aluminum. In the normal operational mode or during maintenance, the composite facing of an aircraft structure may be exposed to foreign object impact from dropped handtools, runway debris, sometimes even from birds and hail stones. The damage is often internal and not visible from the impact side, and may escape inspections. Consequently, there is a greater need for understanding the impact response of composites for better design. A number of works on foreign object impact damage to composites can be found in[4].

Sun and Yang[5] used experimentally measured contact laws to study the impact response of composite laminates. The impacted structure was modelled by finite elements, but the basic principle was the same as that described in earlier works[2, 3]. In this paper we proposed a numerical algorithm that will drastically reduce the computational effort. Also, this method will be very advantageous, if one is interested in parametric studies by changing the impact parameters such as impactor mass, impact velocity and the contact law.

2. AN EFFICIENT NUMERICAL METHOD

Assume that a rigid mass m impacts a structure of any general type with a velocity v_0. We are concerned only with problems wherein the direction of impact is normal to the structure at the point of impact. Reckoning the beginning of contact as time $t = 0$, the equation of the the impactor can be written as

$$m\ddot{x} = -F(t), \tag{1}$$

where x is the displacement of the impactor, and $F(t)$ is the contact force which we are interested in. The double dots in the above equation denote the second derivative with respect to time. Let $w(t)$ be the normal displacement of the structure at the impact point. $w(t)$ can formally be written as

$$w(t) = \int_0^t F(\tau)g(t-\tau)\,\mathrm{d}\tau, \tag{2}$$

where $g(t)$ is the dynamic Green's function[6] for the w displacement. It should be noted that $g(t)$ depends entirely on the properties of the impacted structure. By definition, indentation α is the difference in displacement of the impactor and that of the structure at the impact point. Thus

$$\alpha(t) = x(t) - w(t). \tag{3}$$

The contact law may be of a general form as given below:

$$\begin{aligned} &F = \phi(\alpha) \text{ or } \alpha = \bar{\phi}(F), \alpha \geq \alpha_0 \\ &F = 0,\ \alpha < \alpha_0. \end{aligned} \tag{4}$$

In the above relations α_0 is the permanent indentation, which is zero as long as the indentation does not exceed a critical value. Details on such inelastic contact laws are given in[5]. In practice, eqns (1)–(4) are solved by an iterative scheme. In what follows we suggest an algorithm wherein the iterations of the above nonlinear problem are separated from the structural response computations, thus making the numerical procedures much simpler.

We shall assume that the contact force variation is linear in time during each small increment Δt. Thus

$$F(t) = \sum_{i=0,1,2,\dots} q_i R\langle t - i\Delta t\rangle, \tag{5}$$

where q_i's are the unknowns that decide the contact force history, and $R\langle\ \rangle$ is a function in time defined as follows:

$$\begin{aligned} R\langle t - t_0\rangle &= 0, \quad \leq t \leq t_0 \\ &= (t - t_0)/\Delta t, \quad t_0 \leq t \leq t_0 + \Delta t \\ &= 1, \quad t \rangle t_0 + \Delta t. \end{aligned}$$

Substituting (5) in (2) we get

$$\begin{aligned} w(t) = &\sum_{i=0,1,2,\dots} q_i H\langle t - i\Delta t\rangle \\ &\times \int_{i\Delta t}^{t} R\langle \tau - i\Delta t\rangle g(t-\tau)\,\mathrm{d}\tau, \end{aligned} \tag{6}$$

where $H\langle\ \rangle$ is the Heaviside step function. The above integral can be simplified by the substitution $\theta = \tau - i\Delta t$. Thus

$$\begin{aligned} &\int_{i\Delta t}^{t} R\langle \tau - i\Delta t\rangle g(t-\tau)\,\mathrm{d}\tau \\ &= \int_0^{t-i\Delta t} R\langle\theta\rangle g(t - i\Delta t - \theta)\,\mathrm{d}\theta \\ &= S(t - i\Delta t), \end{aligned}$$

where

$$S(t) = \int_0^t R\langle\tau\rangle g(t-\tau)\,\mathrm{d}\tau. \tag{7}$$

One can see that $S(t)$ is nothing but the response of the structure to the loading $R\langle t\rangle$. So, eqn (6) becomes

$$w(t) = \sum_{i=0,1,2,\dots} q_i H\langle t - i\Delta t\rangle S(t - i\Delta t). \tag{8}$$

Since $S(t)$ depends only on the impacted structure or the target, it is seen that $w(t)$ has been expressed as a linear function of the unknown q_i's.

From eqns (1), (5) and (8) we can write the expressions for the contact force, velocity, and displacements at any time $t = i\Delta t$ as follows.

$$\begin{aligned} F_i &= q_0 + q_1 + q_2 + \dots\dots + q_{i-1} \\ &= F_{i-1} + q_{i-1} \\ v_i &= v_{i-1} - F_{i-1}\Delta t/m - q_{i-1}\Delta t/2m \\ x_i &= x_{i-1} + v_{i-1}\Delta t - F_{i-1}(\Delta t)^2/2m - q_{i-1}(\Delta t)^2/6m \\ w_i &= q_0 S_i + q_1 S_{i-1} + q_2 S_{i-2} + \dots\dots + q_{i-1}S_1. \end{aligned} \tag{9}$$

In the above equations v_i is the velocity of the impactor at $t = i\Delta t$ and $S_i = S(i\Delta t)$. Using (9), eqns (3) and (4) can be written as

$$\alpha_i = x_i - w_i$$

or

$$\bar{\phi}(F_{i-1} + q_{i-1}) = C_i - q_{i-1}\{S_1 + (\Delta t)^2/6m\}, \tag{10}$$

where

$$\begin{aligned} C_i = {} & x_{i-1} + v_{i-1}\Delta t - F_{i-1}(\Delta t)^2/2m \\ & - (q_0 S_i + q_1 S_{i-1} + \dots\dots + q_{i-2}S_2). \end{aligned}$$

Equation (10) is a nonlinear algebraic equation in the unknown q_{i-1} which determines the contact force at $t = i\Delta t$ as $F_i = F_{i-1} + q_{i-1}$. It may be noted that the factor C_i is computed from the impact history up to $t = (i-1)\Delta t$ and hence it is known. Equation (10) can be solved by assuming a starting value for q_{i-1} (usually zero) from which the left hand side can be evaluated. From this an improved q_{i-1} is obtained as

$$q_{i-1} = \{C_i - \bar{\phi}(F_i)\}/\{S_1 + (\Delta t)^2/6m\}.$$

The iteration is continued until the difference between successive values of q_{i-1} is less than a predetermined maximum, say 1% of q_{i-1}. From the final value of q_{i-1} other quantities such as F_i, w_i, x_i, and v_i can be computed using the set of eqns (9). Solving eqn (10) is much easier and faster, especially when the structure has to be modelled by finite elements. In such cases finite element method is used to compute the S–function numerically.

If one is interested in the respones of the structure such as stress, strain, or displacement at a particular point, it can be obtained using the convolution integral as explained below. Let us assume that the strain at a point can be expressed as

$$e(t) = \int_0^t F(\tau) g_e(t-\tau)\,\mathrm{d}\tau,$$

where $g_e(t)$ is the Green's function for the strain. By following the steps used in deriving eqn (8) it can be shown that

$$e(t) = \sum_{i=0,1,2,\dots} q_i H\langle t - i\Delta t\rangle S^e(t - i\Delta t),$$

where

$$S^e(t) = S_i^e = \int_0^t R\langle\tau\rangle g_e(t-\tau)\,\mathrm{d}\tau.$$

Thus, the strain response due to impact is given by

$$\begin{aligned} e(i\Delta t) = e_i = {} & q_0 S_i^e + q_1 S_{i-1}^e + q_2 S_{i-2}^e \\ & + \dots\dots + q_{i-1}S_1^e. \end{aligned}$$

From the above discussions it is evident that the only information about the structure that is needed to solve the impact problem is the function $S(t)$. Once $S(t)$ is generated, it can be used as a numerical

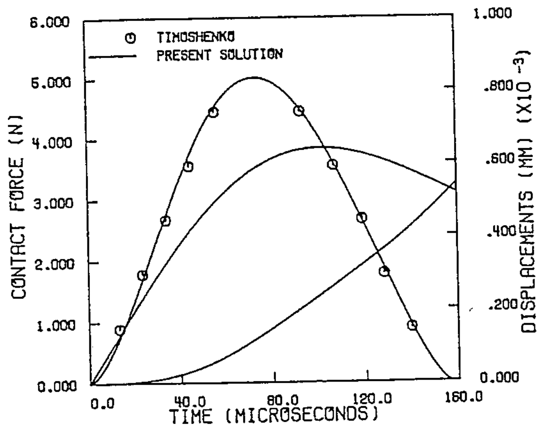

Fig. 1. Contact force and displacements for the Timoshenko problem No. 1[7].

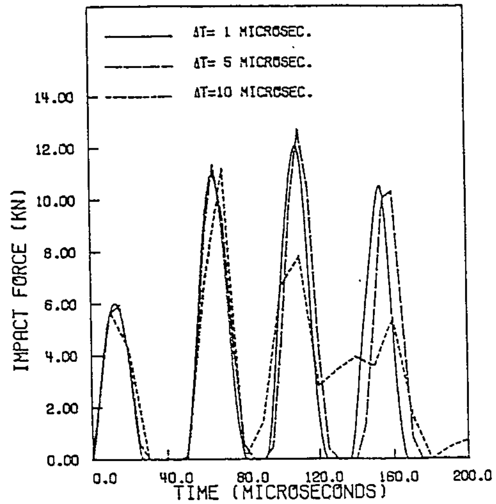

Fig. 3. Effect of Δt on the contact force history.

Green's function to solve the impact problem irrespective of the impact parameters such as impactor mass, impact velocity, and the contact law. Similarly, the function $S^e(t)$ can be used to compute the structural response like strain (stress or displacement) due to impact.

3. NUMERICAL EXAMPLES

In order to validate the above described method several examples were solved. One of them was the Timoshenko impact problem[7] wherein a $1 \times 1 \times 15.35$ cm simply supported steel beam was impacted by a 1 cm radius steel ball with an initial velocity of 1 cm/s. The beam properties were such that the fundamental frequency is 1000 Hz. The same problem was also solved by Sun and Huang[8] using high order beam finite elements. While using the present method the Green's function for the central deflection of the beam was obtained by using the modal expansion method[6]. The time increment Δt was 5 μS. The results shown in Fig. 1 agree very well with Timoshenko's solution and also with that of Sun and Huang[8]. If the mass of the impactor is not small compared to that of the beam, then multiple collisions may occur. In order to show that the proposed method can handle such situations, a second example[9] was also solved. In this example a 2 cm radius steel ball impacts a $1 \times 1 \times 30.7$ cm simply supported steel beam with a velocity of 1 cm/s. A time step of 20 μs was used. The results shown in Fig. 2 have excellent agreement with that given in[9]. It should be mentioned that the contact law used in the above examples was the Hertzian relation for the indentation of half-space by a sphere.

4. EFFECT OF VARYING ΔT

The time increment Δt was varied in several examples and the following were observed. The size of Δt depends upon the number of harmonics and their periods involved in the impact force history and also in the strain (or stress) computed using the convolution integral. The time increment should be small enough to span one half period of the force cycle with sufficient number of steps. For example, in Fig. 2 the force attains the first maximum at about 100 μs and a Δt of 20 μs gives good results. In order to describe this more explicitly another example is chosen wherein multiple impacts with smaller periods occur.

A simply supported beam of dimensions

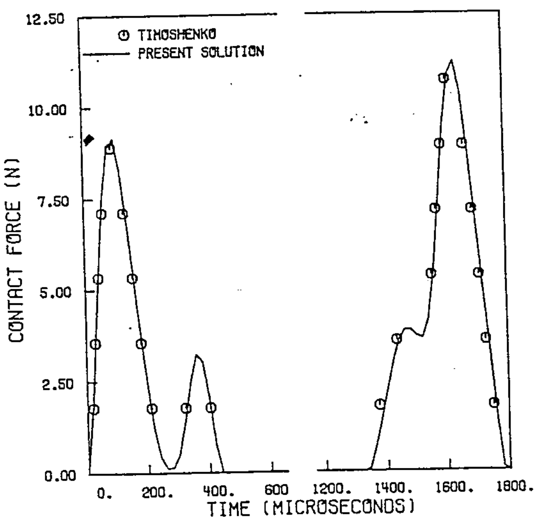

Fig. 2. Contact force history for the Timoshenko problem No. 2[9].

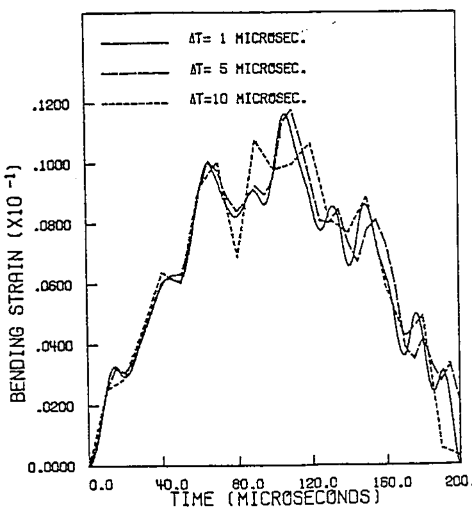

Fig. 4. Effect of Δt on the strain response due to impact.

$25 \times 25 \times 2$ mm is impacted at the center with a ball of mass 33.51 gm. The density of the beam material is 8 gm/cc and the Young's modulus is taken as 2×10^5 MPa. The impact velocity is 10 m/s and the contact law is assumed to be $F = 10^9\alpha^{1.5}\, N/m^{1.5}$. The impact force history and the strain on the outer surface at the center are shown in Figs. 3 and 4 respectively. The impact force consists of several impulses of period about 30 μs. Time steps of 1 and 5 μs give converging results up to a period of about 100 μs. Afterwards there occurs a phase difference, but the magnitudes remain almost the same. A time step of 10 μs predicts the force correctly up to $t = 80$ μs. Thereafter the deviation is significant. The last two collisions merge into a single impact of longer period. Such divergence is conceivable, because the impact event depends upon the motions of both the ball and the beam, and it can be affected by the previous force history. In this example one half period of the impact cycle is about 15 μs, and so a time step of 1 or 2 μs should be good enough to predict the whole impact event. Nevertheless, the strain response (Fig. 4) is good even for $\Delta t = 10$ μs. It may be noted that for larger time increments the information about higher harmonics is missing, but the average strain response is reasonably good.

5. EFFICIENCY OF COMPUTATION

The efficiency of the present method can easily be inferred without considering any particular examples. Assuming the impacted structure is modelled by finite elements, solution of an impact problem by existing methods will take a computational time which is about two to three times of the time required for dynamic analysis by direct integration methods. Also, the time step Δt will usually be much smaller in finite element methods to ensure convergence. Thus the impact problem will require more number of steps for a given time interval. For example, a plate modelled by 16 elements and 400 degrees of freedom requires a central processing time of 400 sec on a CDC 6500 computer for 200 time steps. The same will be about 1000 sec for an impact problem solved by existing methods. Whereas in the present method, once the function $S(t)$ is generated and stored in a magnetic tape, the information can be used for any number of impact problems involving the same structure and the same impact point. The computational time required for computing $S(t)$ will be the same as that required for the dynamic analysis, say 400 sec. With S–function, the impact problem is found to require only about 8 sec of CP time for 200 time steps. To summarize, the CP time required for solving N impact problems by existing methods will be 1000 N seconds, and the same for the proposed method will be (400 + 8N) sec. The computational advantage will increase many times as N increases. This method will be useful in the course of research and development, where one one may have to study the impact of a given structure under various impact conditions.

Acknowledgement—This work was supported by the NASA-Langley Research Center, under Grant No. NAG 1222 with Purdue University.

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