AIAA JOURNAL
Vol. 31, No. 8, August 1993

Passive Damping of Large Space Structures

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The complex stiffness matrix and the mass matrix of a uniaxial bar subjected to constrained layer damping over its entire length are derived exactly by solving the differential equations of motion of the three-layered structure. The stiffness and mass matrices of a bar with segmented damping treatments are obtained by assembling the corresponding matrices for each segment and eliminating the internal nodes using a reduction procedure similar to static condensation. The natural frequencies, mode shapes, and loss factor of a pin-connected truss containing several damped members are computed by three different methods: truss finite element (TFE) method (exact), equivalent beam element (EBE) method, and scaled beam element (SBE) method, each method being more efficient than the preceding one. A 10-bay plane truss is considered as an example to illustrate each method. The EBE method yields very good results, but the savings in computation is not significant. The SBE method reduces the computational effort drastically and gives reasonably approximate results.

I. Introduction

THE space structures of future space stations and other such facilities would be typically latticed, lightweight, and flexible. During regular operation of these space stations, they would be subjected to excitation, causing undesirable low-frequency vibrations. The control of these vibrations is vital for the successful operation of the space structure. In addition to active controls, passive damping techniques can be employed to minimize the mass of components of the active control systems. Constrained layer damping is one of the efficient passive vibration control techniques.[1]

In this paper we have developed a series of analytical and numerical techniques for the analysis of a large space structure (pin-connected truss) subjected to constrained layer damping. Actually, these techniques are equally applicable to any type of passive damping treatment. Figure 1 depicts the hierarchy of models that can be used in analyzing a passively damped large space structure. First, a closed-form expression is derived for computing the complex stiffness matrix and mass matrix of a uniaxial bar subjected to damping treatment along the entire length (Fig. 1a). The problem of segmented treatment on a uniaxial bar is solved by considering the bar as an assemblage of fully treated bars (Fig. 1b). The stiffness and mass matrices of the various uniaxial members are assembled to form the global stiffness and mass matrices of the truss structure (Fig. 1c). The loss factor can be computed using the modal strain energy method.[2] If the truss has a large number of members—which is typical of large space structures—then an equivalent beam or plate model can be derived. There are several approaches for deriving the equivalent continuum model for an undamped space structure, e.g., Sun et al.,[3] Noor et al.,[4] and Lee.[5] All of these methods are based on the assumption that the large space structure has a repeating unit cell. In this paper we have modified the method described by Lee[5] to derive the complex stiffness and mass matrix of an equivalent beam element, which can then be used in the analysis of large structures. Two models, equivalent beam element (EBE) method (Fig. 1d) and scaled beam element (SBE) method (Fig. 1e), are derived. In the EBE method, each unit cell of the truss is modeled as one equivalent beam element, whereas in the SBE method the truss is modeled with beam elements that are generally longer than the unit cell.

Numerical examples are given to illustrate the efficiency of various methods and also their application in optimizing the length of treatment and number of segments for maximum damping.

Received May 5, 1992; revision received Feb. 5, 1993; accepted for publication Feb. 5, 1993.

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II. Damping of a Uniaxial Bar with Full Treatment

Consider the uniaxial bar subjected to constrained layer damping treatment along the entire length (Fig. 2). The equations of motion for the base structure and the constraining layer are:

$$AE\frac{\partial^2 \bar{U}}{\partial X^2} - \rho A\frac{\partial^2 \bar{U}}{\partial t^2} = k(\bar{U} - \bar{V}) \tag{1}$$

$$A_c E_c\frac{\partial^2 \bar{V}}{\partial X^2} - \rho_c A_c\frac{\partial^2 \bar{V}}{\partial t^2} = k(\bar{V} - \bar{U}) \tag{2}$$

where $\bar{U}$ and $\bar{V}$ denote the axial displacements of the base structure and the constraining layer, respectively; A, E, and ρ are the area of cross section, Young's modulus, and the density, respectively, of the base structure; and A_c, E_c, and ρ_c are the corresponding properties of the constraining layer. The viscoelastic damping layer is solely characterized by the shear stiffness per unit length k, which is estimated as Gb/h, where G, b, and h are, respectively, the complex shear stiffness, width of the treatment, and thickness of the viscoelastic damping layer. Assuming the frequency of vibration as ω, the displacements can be written as

$$\bar{U} = U(x)e^{i\omega t} \tag{3}$$

$$\bar{V} = V(x)e^{i\omega t} \tag{4}$$

Substituting from Eqs. (3) and (4) into Eqs. (1) and (2) and solving the set of ordinary differential equations, the solution for the complex displacements $U(x)$ and $V(x)$ can be written as

$$U = \sum_{n=1}^{4} A_n \exp(\lambda_n X) \tag{5}$$

$$V = \sum_{n=1}^{4} B_n \exp(\lambda_n X) \tag{6}$$

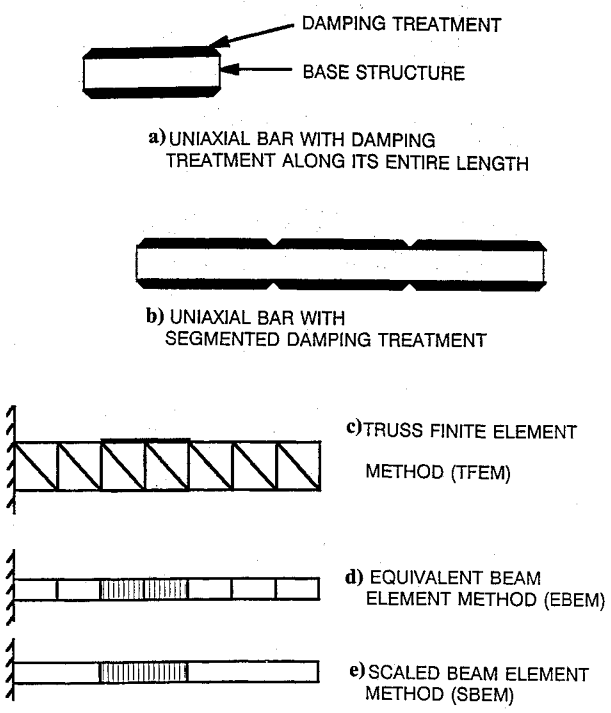

Fig. 1 Hierarchy of models.

where λ_n are the roots of the characteristic equation and the ratio B_n/A_n is given by

$$\frac{B_n}{A_n} = \left(-\frac{1}{k}\right)[AE\lambda_n^2 + (\omega^2\rho A - k)], \qquad (n = 1, \ldots, 4) \quad (7)$$

The arbitrary constants A_n and B_n can be solved in terms of end displacements U_1 and U_2 from the boundary conditions:

$$U(0) = U_1, \qquad U(L) = U_2$$
$$\left.\frac{dV}{dX}\right|_{X=0} = 0, \qquad \left.\frac{dV}{dX}\right|_{X=L} = 0 \quad (8)$$

We have assumed that the constraining layer is not anchored and hence it is stress free at the ends. Thus the solutions for $U(x)$ and $V(x)$ can be expressed in terms of U_1 and U_2 as

$$U(x) = U_1N_1(x) + U_2N_2(x) \quad (9)$$

$$V(x) = V_1H_1(x) + V_2H_2(x) \quad (10)$$

The expressions for λ and the shape functions N_1, N_2, H_1, and H_2 are given in Appendix A.

The stiffness matrices of the base bar, constraining layer, and damping layer are derived by expressing the respective strain energies in terms of the nodal displacements U_1 and U_2. In the following equations, q is the column vector of displacements U_1 and U_2, i.e., $q^T = [U_1\ U_2]$. In the preceding expression and in what follows, a superscript T denotes conjugate transpose of a matrix. The strain energy in the base structure can be expressed as

$$\sqcup_{\text{base}} = \tfrac{1}{2}AE\int_0^L \left|\frac{dU}{dX}\right|^2 dx \quad (11)$$

Substituting for $U(x)$ in terms of q, the strain energy can be derived as

$$\sqcup_{\text{base}} = \tfrac{1}{2}q^T[K_{\text{base}}]q \quad (12)$$

where K_{base} is the stiffness matrix of base structure.

Similarly, for the constraining layer,

$$\sqcup_{\text{cons}} = \tfrac{1}{2}A_cE_c\int_0^L \left|\frac{dV}{dX}\right|^2 dx = \tfrac{1}{2}q^T[K_{\text{cons}}]q \quad (13)$$

where K_{cons} is the stiffness matrix of the constraining layer.

For the viscoelastic layer,

$$\sqcup_{\text{vis}} = \tfrac{1}{2}\,K\int_0^L |U - V|^2\, dx = \tfrac{1}{2}\,q^T[K_{\text{vis}}]q \quad (14)$$

where K_{vis} is the stiffness matrix of the viscoelastic layer.

A similar procedure is used to derive the mass matrices from the expressions for kinetic energy in each of the components. For example, the mass matrix of the base bar can be derived from

$$\tfrac{1}{2}\,\rho A\int_0^L |U|^2\, dx = \tfrac{1}{2}q^T[M_{\text{base}}]q \quad (15)$$

The mass of the viscoelastic layer is assumed to be negligible and its kinetic energy contribution is ignored.

The various stiffness and mass matrices are given in Appendix A.

III. Damping of a Uniaxial Bar with Segmented Treatment

A uniaxial bar with damping treatment applied in discontinuous segments is depicted in Fig. 3. Each segment is considered as an individual fully treated member. For portions without any damping treatment, the stiffness and mass properties of the damping and constraining layers can be taken as zero. The stiffness and mass matrices of all of the segmented parts are calculated as described in Sec. II and then assembled to form the complex stiffness matrix K and mass matrix M. For a bar with N segments, the order of the stiffness or mass matrices will be $(N + 1) \times (N + 1)$. Since no forces are acting at the internal nodes, a static condensation procedure can be used to eliminate those degrees of freedom and reduce the order of the stiffness and mass matrices to 2×2.

The dynamic stiffness matrix D of the bar at a frequency ω is

$$D = [K_r - \omega^2 M] \quad (16)$$

where K_r is the real part of the stiffness matrix K. In Eq. (16), the damping effects are assumed small and ignored. The equations of motion can be arranged as

$$\begin{bmatrix} D_{RR} & D_{RC} \\ D_{CR} & D_{CC} \end{bmatrix} \begin{bmatrix} U_R \\ U_C \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix} \quad (17)$$

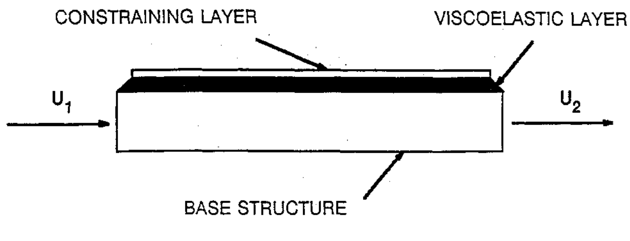

Fig. 2 Constrained layer damping of a uniaxial bar.

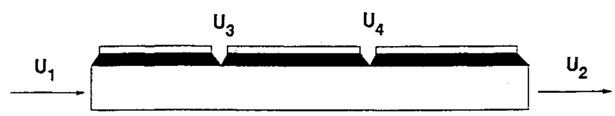

Fig. 3 Segmented damping treatment of a uniaxial bar.

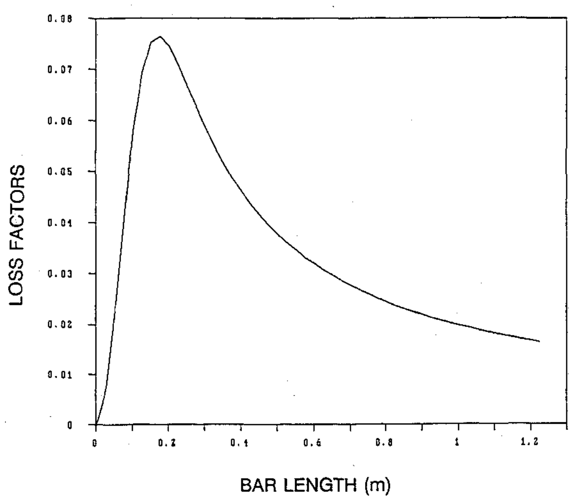

Fig. 4 Loss factors for fully treated bars of various lengths.

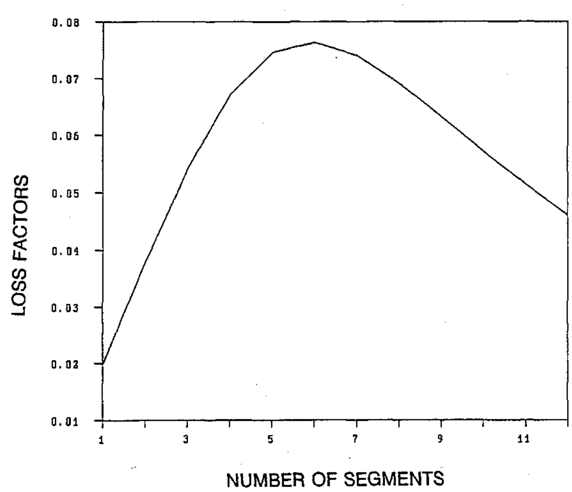

Fig. 5 Loss factor vs number of segments (bar length 1 m).

where the subscripts R and C denote the degrees of freedom to be retained and condensed, respectively, and F is the vector of forces at the end nodes. This gives a relation between the two degrees of freedom to be retained and the condensed degrees of freedom:

$$U_C = -[D_{CC}]^{-1}[D_{CR}]U_R \tag{18}$$

The transformation matrix T that relates the retained and total degrees of freedom can be derived as

$$T^T = [I_2 \quad -D_{RC}D_{CC}^{-1}] \tag{19}$$

Then the reduced 2×2 complex stiffness and mass matrices K^{red} and M^{red} for the segmented bar are

$$K^{\text{red}} = T^T K T \tag{20}$$

$$M^{\text{red}} = T^T M T \tag{21}$$

IV. Numerical Examples for Damping Treatment in Uniaxial Bars

The procedures described in Secs. II and III were applied to a problem of a uniaxial bar fixed at one end and subjected to a unit sinusoidal force at the other end. The frequency of the force ω is assumed to be 13 Hz, which is typical for the truss used in the numerical example later. The equation of motion for this one-degree-of-freedom system is

$$(K_{22}^{\text{red}} - \omega^2 M_{22}^{\text{red}})U_2 = 1 \tag{22}$$

Then the tip displacement U_2 is given by $1/(K_{22}^{\text{red}} - \omega^2 M_{22}^{\text{red}})$. The loss factor is equal to the ratio of the imaginary part and real part of U_2. In Fig. 4 the loss factor is plotted for various lengths of the bar treated with damping layer along their entire length. In Fig. 5 the effect of the number of segments on a bar of a given length is depicted. All of these results were compared with a detailed finite element analysis[6] of the problem, and the comparison was excellent. The material properties used in the examples are given in Table 1.

From Fig. 4 it may be seen that there exists an optimum length of a fully treated bar for maximum damping. For the example considered, the optimum length is about 0.18 m. Figure 5 shows the variation of loss factor with the number of segments of damping treatment on a bar 1 m long. The segments are assumed to be of equal length. It may be noted that the optimum number of segments is six, and hence the length of each segment is 0.167 m. In fact this is consistent with the results shown in Fig. 4, i.e., the optimum segment length in Fig. 5 is approximately equal to the treatment length for maximum damping in Fig. 4.

V. Damping Evaluation Using the Truss Finite Element Method

Consider the problem of a space truss consisting of several pin-connected members subjected to constrained layer damping or any other passive damping devices. The stiffness and mass matrices of these bars in the local coordinate system parallel to the bar can be derived using the methods described in the preceding sections. They can be transformed to the global truss coordinates using the traditional transformation matrices for truss elements.[5] From the real part of the global stiffness matrix and the global mass matrix the undamped natural frequencies and mode shapes can be found. Then the loss factor of the truss corresponding to the nth mode is[2]

$$\eta = \frac{\phi_n^T K'' \phi_n}{\phi_n^T K' \phi_n} \tag{23}$$

where ϕ_n is the nth eigenvector (mode shape), and K' and K'' are the real and imaginary parts of the complex global stiffness matrix.

Numerical results for several examples of treatment in a plane truss are given in Tables 2–4 and are discussed in Sec. VIII.

VI. Equivalent Beam Element Method

In the case of large space structures, one can derive an equivalent continuum model, either a beam or a plate model,

Table 1 Material properties used in present study

	Young's modulus E, GPa	Area of cross section A, m^2	Density ρ, kg/m^2	Shear stiffness k, Pa
Base structure	70	1×10^{-4}	2800	—
Constraining layer	200	2×10^{-5}	8000	—
Viscoelastic layer	—	—	—	8×10^8

Table 2 Comparison of natural frequencies of a truss without damping treatment

	Mode I		Mode II		Mode III	
Method	ω, rad/s	η	ω, rad/s	η	ω, rads/s	η
TFE	72.47	0	369.84	0	705.18	0
EBE	72.65	0	377.63	0	705.64	0
SBE	66.44	0	324.13	0	719.30	0

Table 3 Results for the truss with diagonal members damped

Method	Mode I		Mode II		Mode III	
	ω, rad/s	η	ω, rad/s	η	ω, rads/s	η
TFE	68.44	5.7357*e*-4	366.57	2.9157*e*-3	679.72	5.749*e*-5
EBE	68.61	5.8151*e*-4	374.32	3.0668*e*-3	679.05	6.1435*e*-5
SBE	63.07	7.3708*e*-4	332.95	4.5889*e*-3	714.53	1.2608*e*-3

Table 4 Results for the truss with horizontal members damped

Method	Mode I		Mode II		Mode III	
	ω, rad/s	η	ω, rad/s	η	ω, rads/s	η
TFE	86.68	1.7803*e*-2	416.79	1.1827*e*-2	728.09	1.9606*e*-2
EBE	87.36	1.8018*e*-2	427.49	1.2430*e*-2	716.77	1.9555*e*-2
SBE	79.31	1.7053*e*-2	358.7	0.9704*e*-2	734.36	1.8354*e*-2

for each repeating unit cell of the truss structure. Lee[5] derived the equivalent beam finite element stiffness and mass matrices for the unit cell of a slender space truss. In the present study the preceding method is extended to derive the imaginary part of the stiffness matrix also. These equivalent matrices can then be assembled to form the global stiffness and mass matrices from which the natural frequencies, mode shapes, and damping in the original large space structure can be computed using the method described in Sec. V. A brief description of Lee's[5] method is given next.

In this method a unit cell of the truss is modeled as a single three-dimensional beam finite element having the same length as the unit cell. The beam element has two extreme nodes and six degrees of freedom at each node. The six degrees of freedom are the displacements in the x, y, and z directions and the rotations about the three coordinate axes. The displacements at an arbitrary point in the beam can be obtained based on the assumption that plane sections remain plane. Thus the displacement components at any point on the left/right cross section can be expressed in terms of the nodal displacements and rotations at the left/right nodes of the beam element. Then a transformation matrix T that will relate the displacements of a real joint in the unit cell to the fictitious beam degrees of freedom is derived. The stiffness and mass matrices of each uniaxial member of the unit cell are transformed to the beam coordinates by using the standard transformation and then augmented to obtain corresponding matrices for the beam element. It may be noted that in the present study the stiffness matrices are complex due to the damping treatment.

The savings in the computational effort one can obtain using this method can be expressed in terms of the reduction in the numbers of degrees of freedom from the real structure to the fictitious beam element. Assuming the number of joints in the unit cell as N, the total number of actual degrees of freedom is $3N$, whereas the beam element has 12 degrees of freedom. Hence the relative savings is equal to $[1 - (4/N)]$. For plane trusses the relative savings will be equal to $[1 - (3/N)]$. For the example shown in Fig. 6, $N = 4$ and the relative savings will be equal to 0.25.

The results from this model are compared with the exact results obtained from the truss finite element (TFE) method in Tables 2–4. The agreement in natural frequencies and structural loss factors is found to be excellent. A detailed discussion of the results is presented in Section VIII.

VII. Scaled Beam Element Method

In Sec. VI we noted that the equivalent beam element method reduces the total number of degrees of freedom; however, the reduction is not very significant. Thus there is no advantage in using the method in the case of large space structures. If we can use a beam element that is longer than the unit cell, then the number of elements needed to model the structure can be reduced, resulting in significant savings in computation. Thus the problem reduces to that of deriving

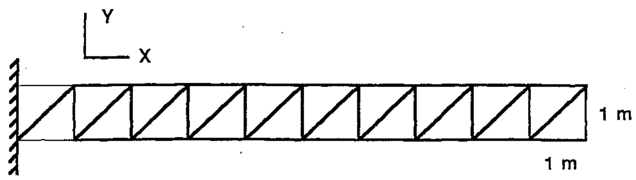

Fig. 6 Ten-bay truss used in the numerical examples.

stiffness and mass matrices of beam elements of arbitrary lengths from those of the equivalent beam derived from the unit cell. One way of accomplishing this objective is to identify a set of beam material and cross-sectional properties from the continuum beam element properties,[3-5] and to use them to derive the stiffness and mass matrices of elements of arbitrary length using the standard finite element derivation procedures. When this method was applied to damping problems, the results were not satisfactory. The method of equivalent beam properties[5] yields reasonably correct values for the real part of the stiffness matrix and the mass matrix because the truss structure behaves more or less like a beam on a global scale. However, when the damping treatments are sparse, the assemblage of treated members does not behave like a beam. Hence the beam properties derived for the imaginary part of the stiffness matrix do not represent the damping effects accurately.

In this study we propose a least-squares method that compares the stiffness matrix of the equivalent beam element with the stiffness matrix of an anisotropic beam finite element.[7] In the present plane truss problem there are 21 stiffness terms in the 6×6 symmetric stiffness matrix of the EBE method, whereas the stiffness matrix of an anisotropic beam element involves six beam properties and the element length as given in Appendix B. Hence, a least-squares solution procedure is used to obtain the six best-fitting properties of the beam from the 21 equations obtained by equating each term of the anisotropic beam stiffness matrix to that of the equivalent beam element. The stiffness matrix of a beam element of arbitrary length can be computed from Eq. (B13) in Appendix B. The same procedure was also applied to the imaginary part of the stiffness matrix. Approximation of the mass matrix was more forgiving than the stiffness calculations. Hence it was assumed that the mass matrix scales linearly with the length of the element, as it does for an anisotropic beam,[7] i.e., $m_{ij} = M_{ij}\ell/L$, where m_{ij} is a term in the mass matrix for element length ℓ, M_{ij} is the corresponding term for the equivalent beam element, and L is the unit cell length.

VIII. Results and Discussion

The material properties used in this study are given in Table 1. The plane truss depicted in Fig. 6 was analyzed with and without any damping treatments.

Two cases of damping treatment were considered: 1) only diagonal members subjected to damping, and 2) only the horizontal members (parallel to the x axis) subjected to damping. The results for natural frequencies and damping were obtained for the first three modes of the truss. The results are presented in Tables 2–4. It should be mentioned that the methods presented in this paper assume the frequency of vibration a priori, and hence an iterative method has to be used to find the correct frequencies for the calculation of stiffness and mass matrices of a damped uniaxial bar. In this iterative method, a frequency is assumed to compute the stiffness and mass matrices of a uniaxial bar, which are used in the truss calculations to determine the natural frequencies of the truss. The calculated natural frequency of the truss is used to correct the uniaxial bar matrices, and the computations were repeated. Usually two or three iterations were needed to estimate the correct natural frequencies. The results were computed using three methods: 1) truss finite element method, which is a full-scale modeling of the truss and yields exact results; 2) equivalent beam element method, where each unit cell is modeled as one beam element; and 3) scaled beam element method, in which the beam element is longer than the unit cell.

The efficiency of computation can be measured in terms of the number of degrees of freedom (DOF) used in each method. For the 10-bay plane truss, the TFE method used 44 DOF, the EBE method used 33 DOF, and the SBE method with four elements used 15 DOF.

From Tables 2–4 it may be seen that the EBE method predicts the natural frequencies accurately, whereas the SBE method has an error of about 10%. In predicting loss factor, the EBE method is very good. The SBE method works reasonably well for the truss with only horizontal members damped, whereas it fails to predict damping for the case when only diagonal members damped. However, the loss factors for the diagonal members damped case (0.06%) is much smaller than the horizontal members damped case (2%). So, in practice, treatment on the top and bottom members is suggested, and the SBE method yields satisfactory results at a much reduced computational effort.

Appendix A: Equations for a Uniaxial Bar with Damping Treatment

In this Appendix the derivations of the solutions for $U(x)$ and $V(x)$, the complex displacements of the base structure and constraining layer, are presented. The derivations of the expressions for the stiffness and mass matrices are also included.

The substitution of the solutions for $U(x)$ and $V(x)$ in the equations of motion gives a characteristic equation whose roots λ are given by

$$\lambda^2 = \frac{-[(\omega^2\rho A - k)A_cE_c + (\omega^2\rho_c A_c - k)AE]}{2AEA_cE_c} \pm \frac{\sqrt{[(\omega^2\rho A - k)A_cE_c + (\omega^2\rho_c A_c - k)A_cE_c]^2 - 4A_cE_cAE(\omega^2\rho A - k)(\omega^2\rho_c A_c - k)}}{2AEA_cE_c} \tag{A1}$$

Using

$$\frac{B_n}{A_n} = R_n = \left(\frac{-1}{k}\right)\left[AE\lambda_n^2 + (\omega^2\rho A - k)\right], \qquad n = 1, \ldots, 4 \tag{A2}$$

The constants in the solution of $U(x)$ and $V(x)$ can be determined in terms of the end displacements U_1 and U_2 by using the boundary conditions in Sec. II and Eq. (8):

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ e^{\lambda_1 L} & e^{\lambda_2 L} & e^{\lambda_3 L} & e^{\lambda_4 L} \\ R_1A_1\lambda_1 & R_2A_2\lambda_2 & R_3A_3\lambda_3 & R_4A_4\lambda_4 \\ R_1A_1\lambda_1 e^{\lambda_1 L} & R_2A_2\lambda_2 e^{\lambda_2 L} & R_3A_3\lambda_3 e^{\lambda_3 L} & R_4A_4\lambda_4 e^{\lambda_4 L} \end{bmatrix}^{-1} \begin{bmatrix} U_1 \\ U_2 \\ 0 \\ 0 \end{bmatrix} \tag{A3}$$

The displacement $U(x)$ is given by

$$U(x) = [e^{\lambda_1 x} \quad e^{\lambda_2 x} \quad e^{\lambda_3 x} \quad e^{\lambda_4 x}]\{A\} \tag{A4}$$

The displacement $V(x)$ is given by

$$V(x) = [R_1e^{\lambda_1 x} \quad R_2e^{\lambda_2 x} \quad R_3e^{\lambda_3 x} \quad R_4e^{\lambda_4 x}]\{A\} \tag{A5}$$

Substituting for $\{A\}$ from Eq. (A3)

$$\{A\} = [T]\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} \tag{A6}$$

and using the expressions for the strain energy and kinetic energy from Sec. II, the various stiffness and mass matrices can be derived as follows.

The stiffness matrix of the base is given by

$$K_{\text{base}} = AE[T]^T[P^{\text{base}}][T] \tag{A7}$$

where

$$P_{ij}^{\text{base}} = \frac{\lambda_i\lambda_j[e^{(\lambda_i+\lambda_j)L} - 1]}{(\lambda_i + \lambda_j)} \qquad i, j = 1, \ldots, 4 \tag{A8}$$

The stiffness matrix of the constraining layer is given by

$$K_{\text{cons}} = A_cE_c[T]^T[P^{\text{cons}}][T] \tag{A9}$$

where

$$P_{ij}^{\text{cons}} = \frac{\lambda_i\lambda_jR_iR_j[e^{(\lambda_i+\lambda_j)L} - 1]}{(\lambda_i + \lambda_j)} \qquad i, j = 1, \ldots, 4 \tag{A10}$$

The stiffness of the viscoelastic layer is given by

$$K_{\text{vis}} = k[T]^T[P^{\text{vis}}][T] \tag{A11}$$

where

$$P_{ij}^{\text{vis}} = \frac{(1 - R_i)(1 - R_j)[e^{(\lambda_i+\lambda_j)L} - 1]}{(\lambda_i + \lambda_j)} \qquad i, j = 1, \ldots, 4 \tag{A12}$$

The mass matrices are derived from the expressions of kinetic energies of the base and constraining layer.

The mass matrix of the base structure is given by

$$M_{\text{base}} = \rho A[T]^T[Q^{\text{base}}][T] \tag{A13}$$

where

$$Q_{ij}^{\text{base}} = \frac{[e^{(\lambda_i+\lambda_j)L} - 1]}{(\lambda_i + \lambda_j)} \qquad i, j = 1, \ldots, 4 \tag{A14}$$

The mass matrix of the constraining layer is given by

$$M_{\text{cons}} = \rho_c A_c [T]^T [Q^{\text{cons}}][T] \tag{A15}$$

where

$$Q_{ij}^{\text{cons}} = \frac{R_i R_j [e^{(\lambda_i \lambda_j)L} - 1]}{\lambda_i + \lambda_j} \qquad i, j = 1, \ldots, 4 \tag{A16}$$

Appendix B: Stiffness Matrix of an Anisotropic Beam Finite Element

In this Appendix the derivation of the stiffness matrix of an anisotropic beam finite element is presented. The method is similar to that given in Ref. 7. Let the constitutive relations of an anisotropic beam be

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{Bmatrix} \epsilon_0 \\ \gamma_0 \\ \kappa \end{Bmatrix} = \begin{Bmatrix} P \\ V \\ M \end{Bmatrix} \tag{B1}$$

$$\epsilon_0 = u_{,x}, \qquad \gamma_0 = \theta + w_{,x}, \qquad \kappa = \theta_{,x} \tag{B2}$$

where u, w, and θ are the axial displacement at the midplane, transverse displacement, and rotation of the cross section; P, V, and M are the axial force resultants, shear force resultant, and the bending moment, respectively; and the subscript x after a comma denotes differentiation with respect to x. The symmetric matrix $[c]$ is a function of the beam material properties and cross-sectional properties. Consider a beam finite element of length L. The beam element has two nodes and six degrees of freedom u_1, w_1, and θ_1 at the first node and u_2, w_2, and θ_2 at the second node. Let the corresponding nodal forces be F_{x1}, F_{z1}, and m_1 and F_{x2}, F_{z2}, and m_2. The average beam deformations, axial strain, shear strain, and curvature at the center of the beam can be expressed in terms of the nodal variables as

$$\epsilon_0 = \frac{u_2 - u_1}{L} \tag{B3}$$

$$\gamma_0 = \frac{\theta_1 + \theta_2}{2} + \frac{w_2 - w_1}{L} \tag{B4}$$

$$\kappa = \frac{\theta_2 - \theta_1}{L} \tag{B5}$$

The average force resultants at the center of the beam are

$$P = \frac{F_{x2} - F_{x1}}{2} \tag{B6}$$

$$V = \frac{F_{z2} - F_{z1}}{2} \tag{B7}$$

$$M = \frac{m_2 - m_1}{2} \tag{B8}$$

Then we require that the average force resultant and the average deformations satisfy the beam constitutive relations (B1). Thus we obtain three equations relating the nodal forces and nodal displacements. Further the nodal forces should satisfy the equilibrium conditions as

$$F_{x1} + F_{x2} = 0 \tag{B9}$$

$$F_{z1} + F_{z2} = 0 \tag{B10}$$

$$m_1 + m_2 - LF_{z2} = 0 \tag{B11}$$

From the six equations a relation between the nodal forces and nodal displacements can be derived as

$$[k][u_1 \quad w_1 \quad \theta_1 \quad u_2 \quad w_2 \quad \theta_2]^T = [F_{x1} \quad F_{z1} \quad m_1 \quad F_{x2} \quad F_{z2} \quad m_2]^T \tag{B12}$$

where the beam element stiffness matrix $[k]$ is given by

$$[k] = \begin{bmatrix} \frac{c_{11}}{l} & \frac{c_{13}}{l} & \left(\frac{c_{12}}{l} - \frac{c_{13}}{2}\right) & -\frac{c_{11}}{l} & -\frac{c_{11}}{l} & \left(-\frac{c_{12}}{l} - \frac{c_{13}}{2}\right) \\ — & \frac{c_{33}}{l} & \left(\frac{c_{23}}{l} - \frac{c_{33}}{2}\right) & -\frac{c_{13}}{l} & -\frac{c_{33}}{l} & \left(-\frac{c_{23}}{l} - \frac{c_{33}}{2}\right) \\ — & — & \left(\frac{c_{22}}{l} + \frac{c_{33} l}{4} - c_{23}\right) & \left(-\frac{c_{12}}{l} + \frac{c_{13}}{2}\right) & \left(-\frac{c_{23}}{l} + \frac{c_{33}}{2}\right) & \left(-\frac{c_{22}}{l} + \frac{c_{33} l}{4}\right) \\ — & — & — & \frac{c_{11}}{l} & \frac{c_{13}}{l} & \left(\frac{c_{12}}{l} + \frac{c_{13}}{2}\right) \\ — & — & \text{(SYMMETRIC)} & — & \frac{c_{33}}{l} & \left(\frac{c_{23}}{l} + \frac{c_{33}}{2}\right) \\ — & — & — & — & — & \left(\frac{c_{22}}{l} + c_{23} + \frac{c_{33} l}{4}\right) \end{bmatrix} \tag{B13}$$

Acknowledgment

This project was supported by a grant from the Technological Research and Development Authority of the State of Florida.

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