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AN EXACT SOLUTION FOR STRESSES IN LAMINATED BEAMS

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ABSTRACT

The corrections to the Euler-Bernoulli displacement field in a laminated beam are expressed in terms of the axial normal stress σ_{xx} such that the stress equilibrium equations and the traction boundary conditions on the bottom surface of the beam are satisfied exactly. From the assumed displacement field, which is exact, expressions for bending moment and axial force resultants are derived as in the classical beam theories. When equated to the applied bending moment and axial force resultants, they yield an integro-differential equation in σ_{xx}, which has a simple recursion solution. The first term of the series solution corresponds to the classical beam theory solution. By comparing the present solution to that of Timoshenko beam theory, a closed-form expression for shear correction factor for laminated beams is derived. Numerical examples are presented for the case of a sandwich beam.

1. INTRODUCTION

Recent developments in advanced fiber and matrix materials, and also in manufacturing processes have made it possible to use fiber composites in a variety of commercial applications. The same thing could be said of sandwich construction also. For example, fiber composites and sandwich construction are used in automobile, marine and aerospace structures, robots and in biomedical devices. They are used not only as laminated plates, but in various shapes and forms. Composite beams are very common in many applications, e.g., automobile suspensions, hip prosthesis, etc. With the increasing use of composites there is a need for simple and efficient analysis procedures for beam like structures.

Composite beams can be analyzed using one of the following methods: (a) Classical and shear deformable beam theories; (b) Two-dimensional elasticity theory; and (c) Finite element methods. The beam theories are well developed and documented. Some of the beam theories can be considered as quasi-elasticity theories because, the assumptions involved are based on elasticity solutions, [1, 2]. More recently Sankar [3] has derived a beam theory for multidirectional laminates from the shear deformable plate theory, which accounts for bending-twisting coupling and thus includes the effect of torsion. Most of the elasticity solutions are based on Fourier series expansion of the loading function, [4, 5, 6] used elasticity solution for a point force on the half-space to derive expressions for stresses in a beam. In the realm of finite elements Sankar [7] developed an offset beam finite element which is convenient in modeling delaminated beams.

The present paper is concerned with exact solutions for stresses in a laminated beam. The corrections to the Euler-Bernoulli displacement field in a laminated beam are expressed in terms of the axial normal stress σ_{xx} such that the stress equilibrium equations and the traction boundary conditions on the bottom surface of the beam are satisfied exactly. From the assumed displacement field, which is exact, expressions for bending moment and axial force resultants are derived as in the classical beam theories. When equated to the applied bending moment and axial force resultants, they yield an integro-differential equation in σ_{xx}, which has a simple recursion solution. The first term of the series solution corresponds to the classical beam theory solution. The results are compared to some available solutions for homogeneous isotropic beams. The present method is also illustrated for the case of a sandwich beam. By comparing the present solution to that of Timoshenko beam theory solution, a closed-form expression for shear correction factor for laminated beams is derived.

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$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \tag{9}$$

The above relations can be integrated to obtain formal expressions for the displacement field in the beam as follows :

$$w(x, z) = w_0(x, z) + I_z \varepsilon_{zz} \tag{10}$$

$$u(x, z) = u_0(x) - z\frac{\partial w_0}{\partial x} + I_z \gamma_{xz} - I_z I_z \frac{\partial \varepsilon_{zz}}{\partial x} \tag{11}$$

When multiple operators are used as in equation (11), we will assume that they are evaluated from right to left. This will avoid the use of parentheses to separate the integral operators.

We will use the stress-strain relations (2) and (3) to replace ε_{zz} and γ_{xz} in equations (10) and (11) by terms containing σ_{xx}, σ_{zz} and τ_{xz} to obtain the following expressions for the displacement field:

$$w(x, z) = w_0(x, z) + I_z (s_{13}\sigma_{xx} + s_{33}\sigma_{zz}) \tag{12}$$

$$u(x, z) = u_0(x) - z\frac{\partial w_0}{\partial x} + I_z\left(\frac{\tau_{xz}}{c_{55}}\right) - I_z I_z\left(s_{13}\frac{\partial \sigma_{xx}}{\partial x} + s_{33}\frac{\partial \sigma_{zz}}{\partial x}\right) \tag{13}$$

The stress equilibrium equations (neglecting body forces) are :

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = 0 \tag{14}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = 0 \tag{15}$$

By integrating the above equations τ_{zx} and σ_{zz} can be expressed in terms of σ_{xx} as follows :

$$\tau_{zx} = -I_z\left(\frac{\partial \sigma_{xx}}{\partial x}\right) \tag{16}$$

$$\sigma_{zz} = I_z I_z\left(\frac{\partial^2 \sigma_{xx}}{\partial x^2}\right) \tag{17}$$

In deriving (16) and (17) we have assumed that no tractions act on the bottom surface of the beam, $z=0$. Thus the external forces can only be applied to the top surface of the beam, $z=h$. The case of a beam loaded at the bottom can be treated in an analogous manner, or by simply shifting the coordinate system. The case where loads act on both top and bottom surfaces can be dealt by superposition.

Substituting from (16) and (17) into equations (12) and (13), the expressions for displacements take the form :

$$w(x, z) = w_0(x) + I_z s_{13}\sigma_{xx} + I_z s_{33} I_z I_z \frac{\partial^2 \sigma_{xx}}{\partial x^2} \tag{18}$$

$$u(x, z) = u_0(x) - z\frac{\partial w_0}{\partial x} - I_z\left(\frac{1}{c_{55}}\right) I_z \frac{\partial \sigma_{xx}}{\partial x} - I_z I_z s_{13}\frac{\partial \sigma_{xx}}{\partial x} - I_z I_z s_{33} I_z I_z \frac{\partial^3 \sigma_{xx}}{\partial x^3} \tag{19}$$

By comparing equations (18) and (19) with (6) and (7) respectively, one can identify the correction terms $u_1(x, z)$ and $w_1(x, z)$. Equations (18) and (19) can be thought of as a set of assumed displacement fields in terms of $u_0(x)$, $w_0(x)$, and $\sigma_{xx}(x, z)$, where σ_{xx} is an arbitrary function continuous in x and piecewise continuous in z. Now we will use the standard procedures employed in beam theories to determine these functions.

Next we derive expressions for normal strains from (18) and (19). By differentiating both sides of (18) and (19) with respect to z and x respectively, we obtain

$$\varepsilon_{zz} = s_{13}\sigma_{xx} + L_1\left(\frac{\partial^2 \sigma_{xx}}{\partial x^2}\right) \tag{20}$$

$$(A, B, D) = \int_0^h \bar{c}_{11} (1, z, z^2)\, dz \tag{38}$$

The above definition of the laminate stiffness coefficients is different from that of traditional composite literature where the midplane of the beam is considered as the reference plane. Thus the coupling coefficient B does not vanish for beams which are symmetric about the midplane. Equations (32) and (33) can be written conveniently as

$$\begin{pmatrix} N_x \\ M_x \end{pmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{pmatrix} \dfrac{\partial u_0}{\partial x} \\ -\dfrac{\partial^2 w_0}{\partial x^2} \end{pmatrix} - \begin{pmatrix} N_{xc} \\ M_{xc} \end{pmatrix} \tag{39}$$

The terms that N_{xc} and M_{xc} denote can be obtained by comparing (39) with (32) and (33) :

$$N_{xc} = (M_2 - M_1) \frac{\partial^2 \sigma_{xx}}{\partial x^2} + M_3 \frac{\partial^4 \sigma_{xx}}{\partial x^4} \tag{40}$$

$$M_{xc} = (K_2 - K_1) \frac{\partial^2 \sigma_{xx}}{\partial x^2} + K_3 \frac{\partial^4 \sigma_{xx}}{\partial x^4} \tag{41}$$

Equation (39) can be solved for the reference plane strain and curvature to obtain

$$\begin{pmatrix} \dfrac{\partial u_0}{\partial x} \\ -\dfrac{\partial^2 w_0}{\partial x^2} \end{pmatrix} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{B} & \bar{D} \end{bmatrix} \begin{pmatrix} N_x + N_{xc} \\ M_x + M_{xc} \end{pmatrix} \tag{42}$$

where

$$\begin{bmatrix} \bar{A} & \bar{B} \\ \bar{B} & \bar{D} \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix}^{-1} \tag{43}$$

Substituting the results for the axial strain and curvature from (42) into (27) we obtain an integro-differential equation in σ_{xx} as follows :

$$\sigma_{xx} = \bar{c}_{11} [(\bar{A} + z\bar{B})\ (N_x + N_{xc}) + (\bar{B} + z\bar{D})(M_x + M_{xc})] + (\bar{c}_{13} L_1 - \bar{c}_{11} L_2) \frac{\partial^2 \sigma_{xx}}{\partial x^2} - \bar{c}_{11} L_3 \frac{\partial^4 \sigma_{xx}}{\partial x^4} \tag{44}$$

It must be reminded that N_{xc} and M_{xc} are functions of σ_{xx} as given in equations (40) and (41). For the sake of convenience equation (44) can be written in the following form :

$$\sigma_{xx} = f(z)\, N_x + g(z)\, M_x + I_1 \frac{\partial^2 \sigma_{xx}}{\partial x^2} + I_2 \frac{\partial^4 \sigma_{xx}}{\partial x^4} \tag{45}$$

where $f(z)$ and $g(z)$ can be identified with the classical beam theory solution, and the definition of integral operators I_1 and I_2 can be obtained by comparing (45) with (44) :

$$f(z) = \bar{c}_{11} (\bar{A} + z\bar{B}) \tag{46}$$

$$g(z) = \bar{c}_{11} (\bar{B} + z\bar{D}) \tag{47}$$

$$I_1(.) = \bar{c}_{11} (\bar{A} + z\bar{B})(M_2 - M_1)(.) + \bar{c}_{11} (\bar{B} + z\bar{D})(K_2 - K_1)(.) + (\bar{c}_{13} L_1 - \bar{c}_{11} L_2)(.) \tag{48}$$

$$I_2(.) = \bar{c}_{11} (\bar{A} + z\bar{B}) M_3(.) + \bar{c}_{11} (\bar{B} + z\bar{D}) K_3(.) - \bar{c}_{11} L_3(.) \tag{49}$$

$$\frac{\partial^2 w_0}{\partial x^2} + \overline{D}\, M_x + \overline{B}\, N_{xc} + \overline{D}\, M_{xc} = 0 \qquad (65)$$

The expressions for N_{xc} and M_{xc} can be found in (40) and (41). We will assume that M_x^{iv} and its higher derivatives are equal to zero, and use the beam theory expression for σ_{xx}, i.e., $\sigma_{xx} = \overline{c}_{11}\,(\overline{B} + z\,\overline{D})\,M_x$, in equations (40) and (41) and substitute into (65) to obtain

$$\frac{\partial^2 w_0}{\partial x^2} + \overline{D}\, M_x + \left[\overline{B}\,(M_2 - M_1) + \overline{D}\,(K_2 - K_1)\right]\overline{c}_{11}\,(\overline{B} + z\,\overline{D})\,\frac{\partial^2 M_x}{\partial x^2} = 0 \qquad (66)$$

Comparing (63) and (66) one can readily obtain an expression for the shear correction factor for a laminated beam as

$$\kappa^{-1} = A_{55}\left[\overline{B}\,(M_1 - M_2) + \overline{D}\,(K_1 - K_2)\right]\overline{c}_{11}\,(\overline{B} + z\,\overline{D}) \qquad (67)$$

It should be remembered that K_1, K_2, M_1 and M_2 are all integral operators acting on $\overline{c}_{11}\,(\overline{B} + z\,\overline{D})$. For a homogeneous beam the integrations are much simpler, and we obtain

$$\kappa^{-1} = 1.2\left(1 + c_{55}\, s_{13} - \frac{c_{13}\, s_{33}\, c_{55}}{c_{11}}\right) \qquad (68)$$

For an isotropic beam under plane stress parallel to the xz-plane the above expression reduces to $\kappa = 0.8333\,(1 + \nu)$, and for the case of plane strain we obtain $\kappa = 0.8333/(1 - \nu)$. If the Poisson's ratio is neglected ($c_{13} = s_{13} = 0$), then we obtain the standard result $\kappa = 0.8333$.

4. RESULTS AND DISCUSSION

It is interesting to note that similar solutions for isotropic rectangular beams were given by Donnell [10], Boley and Tolins [8] and Gatewood and Dale [11]. Donnell used an iterative approach starting form the beam theory solutions. Boley and Tolins expanded the Airy stress function in a series form, and obtained a recursion relation between the functions. Gatewood and Dale modified the Fourier series solution to obtain similar results. All the above methods are not suitable for laminated beams. The present method provides a convenient closed-form expressions for stresses in a laminated beam. Like the other methods, the present method assumes M_x, N_x, and their derivatives are continuous. Thus the solutions will not be accurate near the supports and also where the loads are discontinuous as in the case of point loading or step loading. Explicit expressions can be derived for homogeneous beams. For the case of isotropic beams the functions ϕ are independent of the elastic constants. A few of the functions are given below in terms of $\zeta = z/h$:

$$\phi_0 = \frac{1}{h^2}(12\,\zeta - 6) \qquad (69)$$

$$\phi_2 = \left(\frac{1}{5} - \frac{12}{5}\zeta + 6\,\zeta^2 - 4\,\zeta^3\right) \qquad (70)$$

$$\phi_4 = h^2\left(\frac{71}{2100} - \frac{47}{350}\zeta - \frac{\zeta^2}{5} + \frac{4}{5}\zeta^3 - \frac{3}{4}\zeta^4 + \frac{3}{10}\zeta^5\right) \qquad (71)$$

$$\phi_6 = h^4\left(\frac{79}{63000} + \frac{4\,\zeta}{7875} - \frac{71\,\zeta^2}{2100} + \frac{47\,\zeta^3}{1050} + \frac{\zeta^4}{40} - \frac{3}{50}\zeta^5 + \frac{\zeta^6}{30} - \frac{\zeta^7}{105}\right) \qquad (72)$$

It should be noted that $z = 0$ corresponds to the bottom side of the beam. The first three functions agree with that given by Boley and Tolins [8]. The expressions given in Gatewood and Dale [11] for the coefficient of sixth and eighth derivatives of M_x were found to be in error. The expression given by equation (72) was also verified by numerical integration and found to be correct.

For the purpose of illustration the functions ϕ_n and their integrals for a sandwich beam are presented in this paper. The Young's moduli of the face sheet and core materials are assumed to be 100 GPa and 10 GPa respectively. The Poisson ratios of both materials is equal to 0.25. The thickness of each face sheet is 1 mm and that of the core is 8 mm. The functions

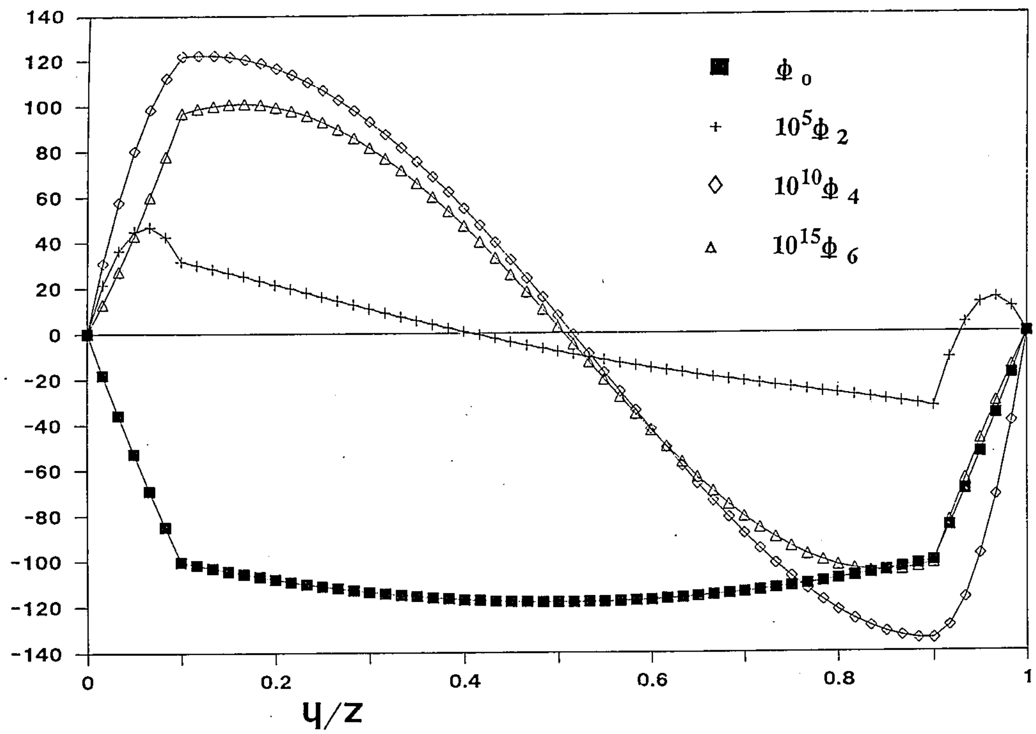

Figure 4. Distribution of ϕ_n's in a sandwich beam

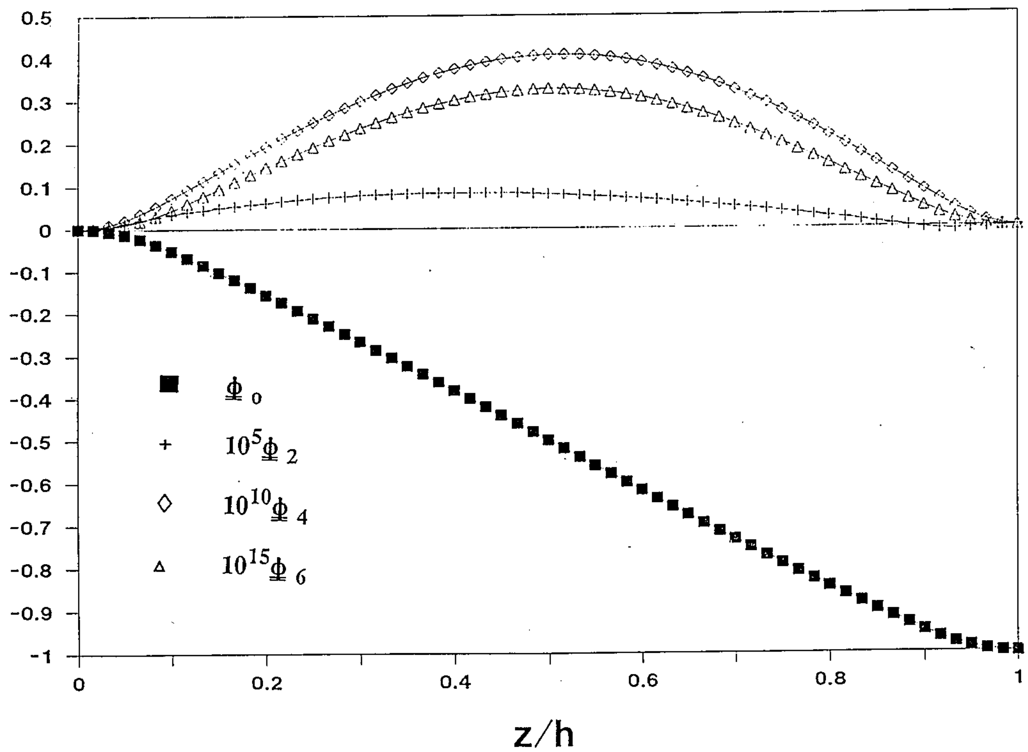

Figure 5. Distribution of ϕ_n's in a sandwich beam