# Non-Singular Term Effect for the Inclined Crack Extension in Anisotropic Solids under Uniaxial Loading

Won-Kyun Lim and Bhavani V. Sankar

*Journal of Composite Materials* 2002; 36; 2031
DOI: 10.1177/0021998302036017251

# Non-Singular Term Effect for the Inclined Crack Extension in Anisotropic Solids under Uniaxial Loading

WON-KYUN LIM*
*Department of Mechanical Engineering*
*Myongji University, Yongin*
*Kyonggi-do 449-728, South Korea*

BHAVANI V. SANKAR
*Department of Aerospace Engineering*
*Mechanics & Engineering Science*
*University of Florida, Gainesville*
*Florida 32611, USA*

(Received September 1, 2001)
(Revised February 4, 2002)

**ABSTRACT:** The problem of predicting crack propagation in anisotropic solids, which is a subject of considerable practical importance, is examined by carrying out the analysis on anisotropic solids with an inclined crack subject to uniaxial loading. By deriving the subsequent term of the series expansion for crack tip stresses in anisotropic materials, its effects on the hoop stresses near the crack tip and predicted crack propagation direction are evaluated. In order to determine the direction of crack extension, the normal stress ratio theory is employed. The theoretical analysis is performed for a wide range of the anisotropic material properties. Based on this failure criterion, it is shown that the second order term in the series expansion is essential for accurate determination of crack growth direction in anisotropic solids.

**KEY WORDS:** crack extension angle, anisotropic solids, uniaxial load, second order stress term, mixed mode crack.

## INTRODUCTION

COMPOSITE MATERIALS HAVE been increasingly used in aerospace and automotive applications. The use of composite materials is very attractive because of their outstanding strength, stiffness, and light-weight properties. The increasing use of advanced composite materials in structural applications has considerably renewed the

*Author to whom correspondence should be addressed. E-mail: limwk@mju.ac.kr



0021-9983/02/17 2031–14 $10.00/0 DOI: 10.1106/002199802026251

interest in solutions to problems in anisotropic elasticity. Problems which have received much attention in this regard include those with cracks, and concepts of fracture mechanics, which have been used for analyzing cracked isotropic bodies, have been extended to treat material anisotropy.

The failure analysis and strength evaluation of composite materials with a notch or a crack is a critical issue in the design and assessment of composite structures. Failure analysis of structural components within the framework of fracture mechanics is based on the initiation and propagation of initial defects such as notch and crack. In designing against fracture in anisotropic composite materials, the prediction of crack initiation and growth direction are of great importance. Fracture problems of anisotropic materials are considerably more complicated than the isotropic case. The direction and load level at which a crack propagates is a function of the stress intensity factor, the crack orientation, the material stiffness properties, and the material strength properties.

The general solution to the problem of cracks in anisotropic materials was obtained by Sih and Liebowitz [1,2]. They found that the stresses at the crack tip have an inverse square root singularity as in classical crack embedded into homogeneous solid. Therefore, it is generally accepted that the elastic stresses and displacements near a crack tip in anisotropic materials can be adequately approximated with sufficient accuracy by the singular expression. In many cases, however subsequent terms of the series expansion are quantitatively significant, and so the evaluation of such terms and their effect on the predicted crack growth direction in anisotropic materials should be considered.

Recently Yang and Yuan [3,4] investigated the effects of higher-order terms in the expression for crack-tip stress field in anisotropic materials. They derived the second and third terms of the crack-tip stress field as additional parameters in characterizing the behavior of the crack. In the previous works of the present authors [5,6], the influence of subsequent terms on predicting Mode I crack propagation angle in anisotropic material was investigated.

In this paper we extend the analysis employed in our previous works to treat the uniaxially loaded anisotropic plate with an inclined crack. We give the correct form of the second-order stress term in the asymptotic expansion for the inclined crack in anisotropic materials. An infinite sheet geometry is used in order to examine the fundamental problem of mixed mode fracture under uniaxial tensile load. The purpose of this research is to show that the direction of crack initiation in plane cracked anisotropic bodies may be significantly affected by the second term. It was shown that although the second term is independent of radial distance, the coefficient of that term depends on the anisotropic material properties, the inclined crack angle, and the applied load.

## PLANE ANISOTROPIC ELASTICITY

The constitutive relations for a homogeneous anisotropic elastic material in plane stress can be written as

$$\begin{aligned}\varepsilon_{xx} &= a_{11}\sigma_{xx} + a_{12}\sigma_{yy} + a_{16}\tau_{xy}\\ \varepsilon_{yy} &= a_{12}\sigma_{xx} + a_{22}\sigma_{yy} + a_{26}\tau_{xy}\\ \gamma_{xy} &= a_{16}\sigma_{xx} + a_{26}\sigma_{yy} + a_{66}\tau_{xy}\end{aligned} \tag{1}$$

where $(\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy})$ and $(\sigma_{xx}, \sigma_{yy}, \tau_{xy})$ are strains and stresses, respectively, and constants $a_{ij}(i,j = 1,2,6)$ are the elastic compliances of the material. These compliances may be given in terms of engineering material constants.

By introducing Airy's stress function $U(x, y)$, Lekhnitskii [7] has shown that the governing equation for a two-dimensional problem of anisotropic elasticity is

$$a_{22}\frac{\partial^4 U}{\partial x^4} - 2a_{26}\frac{\partial^4 U}{\partial x^3 \partial y} + (2a_{12} + a_{66})\frac{\partial^4 U}{\partial x^2 \partial y^2} - 2a_{16}\frac{\partial^4 U}{\partial x \partial y^3} + a_{11}\frac{\partial^4 U}{\partial y^4} = 0 \tag{2}$$

The fundamental solution of Equation (2) for anisotropic 2-D elastic body can be written in terms of complex variables as

$$U(x, y) = 2\mathrm{Re}[U_1(z_1) + U_2(z_2)] \tag{3}$$

where Re denotes the real part of a complex function, $U_1(z_1)$ and $U_2(z_2)$ are stress function of complex variable $z_j = x + s_j y (j = 1, 2)$ and $s_j (j = 1, 2)$ are the roots of the following characteristic equation:

$$a_{11}s^4 - 2a_{16}s^3 + (2a_{12} + a_{66})s^2 - 2a_{26}s + a_{22} = 0 \tag{4}$$

Introduce $\phi(z_1) = dU_1(z_1)/dz_1$ and $\psi(z_2) = dU_2(z_2)/dz_2$. Then by inserting these equations into the relation between stress function and the stresses, the general equations for the stress components in terms of two functions $\phi(z_1)$ and $\psi(z_2)$, can be expressed as

$$\begin{aligned} \sigma_{xx} &= 2\mathrm{Re}[s_1^2\phi'(z_1) + s_2^2\psi'(z_2)] \\ \sigma_{yy} &= 2\mathrm{Re}[\phi'(z_1) + \psi'(z_2)] \\ \tau_{xy} &= -2\mathrm{Re}[s_1\phi'(z_1) + s_2\psi'(z_2)] \end{aligned} \tag{5}$$

where $\phi'(z_1)$ and $\psi'(z_2)$ denote differentiation with the respect complex variables $z_1$ and $z_2$. Substituting the values $\sigma_{xx}, \sigma_{yy}$ and $\tau_{xy}$ from Equation (5) into Equation (1), and by integration, the general equations for the displacements $u(x, y)$ and $v(x, y)$ can be expressed as

$$\begin{aligned} u(x, y) &= 2\mathrm{Re}[p_1\phi(z_1) + p_2\psi(z_2)] \\ v(x, y) &= 2\mathrm{Re}[q_1\phi(z_1) + q_2\psi(z_2)] \end{aligned} \tag{6}$$

To simplify the writing of these equations, the following abbreviations were used

$$\begin{aligned} p_1 &= a_{11}s_1^2 + a_{12} - a_{16}s_1, \quad p_2 = a_{11}s_2^2 + a_{12} - a_{16}s_2 \\ q_1 &= \frac{a_{12}s_1^2 + a_{22} - a_{26}s_1}{s_1}, \quad q_2 = \frac{a_{12}s_2^2 + a_{22} - a_{26}s_2}{s_2} \end{aligned} \tag{7}$$

In orthotropic materials of elastic symmetry, $a_{16} = a_{26} = 0$ and the characteristic equation of (Equation (4)) can be simplified as

$$a_{11}s^4 + (2a_{12} + a_{66})s^2 + a_{22} = 0 \tag{8}$$

It can be shown that for anisotropic materials, Equation (8) has only complex roots, and they are distinct. In addition, because the coefficients of Equation (8) are all real constants, the roots are in complex conjugate pairs. The four roots are denoted by

$$
\begin{aligned}
s_1 &= \sqrt{\frac{\alpha_0-\beta_0}{2}} + i\sqrt{\frac{\alpha_0+\beta_0}{2}} = \alpha_1 + i\beta_1 \\
s_2 &= -\sqrt{\frac{\alpha_0-\beta_0}{2}} + i\sqrt{\frac{\alpha_0+\beta_0}{2}} = \alpha_2 + i\beta_2 \\
s_3 &= \bar{s}_1, \quad s_4 = \bar{s}_2
\end{aligned}
\tag{9}
$$

where $\alpha_0 = \sqrt{a_{22}/a_{11}} = \sqrt{E_{11}/E_{22}}$ and $\beta_0 = (a_{66}/2 + a_{12})/a_{11} = E_{11}/2\mu_{12} - \nu_{12}$.

## CRACK TIP STRESS FIELDS INCLUDING SUBSEQUENT TERM IN ANISOTROPIC MATERIAL

In order to derive the analytic functions, $\phi$ and $\psi$ for an inclined crack in an infinite anisotropic plate under uniaxial loading, we consider an elliptical hole in an infinite plate under tension. Savin [8] has outlined analytic functions for an elliptical hole in a plate which is subjected to uniaxial stress at an angle $\alpha$ with $x$-axis. By substituting zero for the minor axis of the elliptical hole, i.e., $b=0$ and simplifying, the analytic functions for an inclined crack under biaxial loading shown in Figure 1 can be expressed as

$$
\begin{aligned}
\phi(z_1) &= \frac{s_2\sigma^\infty \sin^2\alpha + \sigma^\infty \sin\alpha\cos\alpha}{2(s_1-s_2)}\left[z_1 - \sqrt{z_1^2-a^2}\right] + B^* z_1 \\
\psi(z_2) &= -\frac{s_1\sigma^\infty \sin^2\alpha + \sigma^\infty \sin\alpha\cos\alpha}{2(s_1-s_2)}\left[z_2 - \sqrt{z_2^2-a^2}\right] + (B'^* + iC'^*) z_2
\end{aligned}
\tag{10}
$$

where $B^*, B'^*$ and $C'^*$ are real constants computed from material properties and external loading, and are defined as

$$
B^* = \sigma^\infty \frac{\cos^2\alpha + (\alpha_2^2 + \beta_2^3)\sin^2\alpha + \alpha_2 \sin 2\alpha}{2[(\alpha_2-\alpha_1)^2 + (\beta_2^2-\beta_1^2)]}
$$

$$
B'^* = \sigma^\infty \frac{[(\alpha_1^2-\beta_1^2) - 2\alpha_1\alpha_2]\sin^2\alpha - \cos^2\alpha - \alpha_2 \sin 2\alpha}{2[(\alpha_2-\alpha_1)^2 + (\beta_2^2-\beta_1^2)]}
$$

$$
\begin{aligned}
C'^* = \sigma^\infty \Bigg\{ &\frac{(\alpha_1-\alpha_2)\cos^2\alpha + [\alpha_2(\alpha_1^2-\beta_1^2) - \alpha_1(\alpha_2^2-\beta_2^2)]\sin^2\alpha}{2\beta_2[(\alpha_2-\alpha_1)^2 + (\beta_2^2-\beta_1^2)]} \\
&+ \frac{[(\alpha_1^2-\beta_1^2) - (\alpha_2^2-\beta_2^2)]\sin\alpha\cos\alpha}{2\beta_2[(\alpha_2-\alpha_1)^2 + (\beta_2^2-\beta_1^2)]} \Bigg\}
\end{aligned}
$$

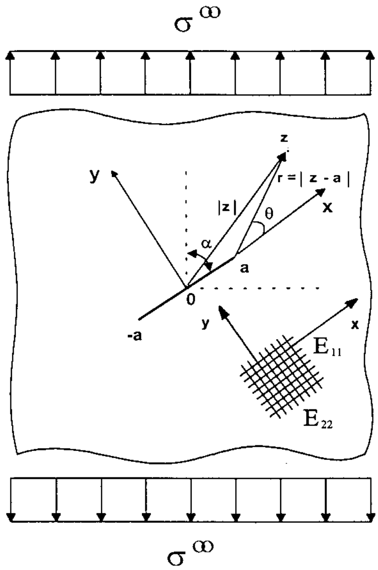


***Figure 1.*** *Inclined crack in an anisotropic plate under uniaxial load.*

The first derivatives of functions $\phi(z_1)$ and $\psi(z_2)$ are given as

$$\phi'(z_1) = \frac{s_2\sigma^\infty \sin^2\alpha + \sigma^\infty \sin\alpha\cos\alpha}{2(s_1 - s_2)}\left[1 - \frac{z_1}{\sqrt{z_1^2 - a^2}}\right] + B^* z_1$$
$$\psi'(z_2) = -\frac{s_1\sigma^\infty \sin^2\alpha + \sigma^\infty \sin\alpha\cos\alpha}{2(s_1 - s_2)}\left[1 - \frac{z_2}{\sqrt{z_2^2 - a^2}}\right] + (B'^* + iC'^*) z_2 \tag{11}$$

Calculation may be facilitated by the use of coordinate $\zeta_j$ originating at crack tip.

$$z_j - a = \zeta_j = r(\cos\theta + s_j \sin\theta), \quad z_j = x + s_j y \qquad (j = 1, 2) \tag{12}$$

Thus, $\phi'(z_1)$ and $\psi'(z_2)$ are given in terms of complex variable $\zeta_j$ $(j = 1, 2)$ as

$$\phi'(\zeta_1) = \frac{s_2\sigma^\infty \sin^2\alpha + \sigma^\infty \sin\alpha\cos\alpha}{2(s_1 - s_2)}\left[1 - \frac{(\zeta_1 + a)}{\sqrt{(\zeta_1^2 + 2a\zeta_1)}}\right] + B^*$$
$$\psi'(\zeta_2) = -\frac{s_1\sigma^\infty \sin^2\alpha + \sigma^\infty \sin\alpha\cos\alpha}{2(s_1 - s_2)}\left[1 - \frac{(\zeta_2 + a)}{\sqrt{(\zeta_2^2 + 2a\zeta_2)}}\right] + (B'^* + iC'^*) \tag{13}$$

where $0 < |\zeta_j| = r \ll 1, (j = 1, 2)$. Expanding the expression inside the bracket on the right side of Equation (13) as a power series, the equation can be written as

$$\phi'(\zeta_1)\} = -\frac{\sigma^\infty s_2 \sin^2\alpha + \sigma^\infty \sin\alpha\cos\alpha}{2(s_1 - s_2)} \left\{ \frac{1}{\sqrt{2}} \left[ \left(\frac{\zeta_1}{a}\right)^{-1/2} + \frac{3}{4}\left(\frac{\zeta_1}{a}\right)^{1/2} - \frac{5}{32}\left(\frac{\zeta_1}{a}\right)^{3/2} + \cdots \right] \right\}$$
$$+ \left(B^* + \frac{\sigma^\infty s_2 \sin^2\alpha + \sigma^\infty \sin\alpha\cos\alpha}{2(s_1 - s_2)}\right)\left(\frac{\zeta_1}{a}\right)^0$$
$$\psi'(\zeta_2)\} = \frac{\sigma^\infty s_1 \sin^2\alpha + \sigma^\infty \sin\alpha\cos\alpha}{2(s_1 - s_2)} \left\{ \frac{1}{\sqrt{2}} \left[ \left(\frac{\zeta_2}{a}\right)^{-1/2} + \frac{3}{4}\left(\frac{\zeta_2}{a}\right)^{1/2} - \frac{5}{32}\left(\frac{\zeta_2}{a}\right)^{3/2} + \cdots \right] \right\}$$
$$+ \left(B'^* + iC'^* + \frac{\sigma^\infty s_1 \sin^2\alpha + \sigma^\infty \sin\alpha\cos\alpha}{2(s_1 - s_2)}\right)\left(\frac{\zeta_2}{a}\right)^0 \qquad (14)$$

Ignoring the higher order terms of $\zeta_1$ and $\zeta_2$ except the terms containing $\zeta_j^{-1/2}$ and $\zeta_j^0$ in Equation (14), the functions of $\phi'(z_1)$ and $\psi'(z_2)$ can be simplified as

$$\phi'(\zeta_1) \cong -\frac{\sigma^\infty s_2 \sin^2\alpha + \sigma^\infty \sin\alpha\cos\alpha}{2\sqrt{2}(s_1 - s_2)} \left[\left(\frac{\zeta_1}{a}\right)^{-1/2}\right] + \left(B^* + \frac{\sigma^\infty s_2 \sin^2\alpha + \sigma^\infty \sin\alpha\cos\alpha}{2(s_1 - s_2)}\right)$$
$$\psi'(\zeta_2) \cong \frac{\sigma^\infty s_1 \sin^2\alpha + \sigma^\infty \sin\alpha\cos\alpha}{2\sqrt{2}(s_1 - s_2)} \left[\left(\frac{\zeta_2}{a}\right)^{-1/2}\right] + \left(B' + iC'^* - \frac{\sigma^\infty s_1 \sin^2\alpha + \sigma^\infty \sin\alpha\cos\alpha}{2(s_1 - s_2)}\right) \qquad (15)$$

Thus, by substituting Equation (15) into Equation (5), the expressions for the near crack tip stresses including the second-order term can be obtained as

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \mathrm{Re}\left[\frac{s_1 s_2}{(s_1 - s_2)}\left(\frac{s_2}{\sqrt{\cos\theta + s_2\sin\theta}} - \frac{s_1}{\sqrt{\cos\theta + s_1\sin\theta}}\right)\right]$$
$$+ \frac{K_{II}}{\sqrt{2\pi r}} \mathrm{Re}\left[\frac{1}{(s_1 - s_2)}\left(\frac{s_2^2}{\sqrt{\cos\theta + s_2\sin\theta}} - \frac{s_1^2}{\sqrt{\cos\theta + s_1\sin\theta}}\right)\right]$$
$$+ \sigma^\infty \mathrm{Re}[(\cos\alpha + s_1\sin\alpha)(\cos\alpha + s_2\sin\alpha)]$$
$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \mathrm{Re}\left[\frac{1}{(s_1 - s_2)}\left(\frac{s_1}{\sqrt{\cos\theta + s_2\sin\theta}} - \frac{s_2}{\sqrt{\cos\theta + s_1\sin\theta}}\right)\right] \qquad (16)$$
$$+ \frac{K_{II}}{\sqrt{2\pi r}} \mathrm{Re}\left[\frac{1}{(s_1 - s_2)}\left(\frac{1}{\sqrt{\cos\theta + s_2\sin\theta}} - \frac{1}{\sqrt{\cos\theta + s_1\sin\theta}}\right)\right]$$
$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \mathrm{Re}\left[\frac{s_1 s_2}{(s_1 - s_2)}\left(\frac{1}{\sqrt{\cos\theta + s_1\sin\theta}} - \frac{1}{\sqrt{\cos\theta + s_2\sin\theta}}\right)\right]$$
$$+ \frac{K_{II}}{\sqrt{2\pi r}} \mathrm{Re}\left[\frac{1}{(s_1 - s_2)}\left(\frac{s_1}{\sqrt{\cos\theta + s_1\sin\theta}} - \frac{s_2}{\sqrt{\cos\theta + s_2\sin\theta}}\right)\right]$$

where $K_I = \sigma^\infty\sqrt{\pi a}(1 - \cos 2\alpha)/2$ and $K_{II} = (\sigma^\infty\sqrt{\pi a})\sin 2\alpha/2$. It can be seen that the second-order term in stresses appears only the in the $\sigma_{xx}$ stress component.

## CRACK EXTENSION DIRECTION IN ANISOTROPIC MATERIALS

We employ the normal stress ratio theory to determine values for the direction of initial crack extension. This criterion, proposed by Buczek and Herakovich [9], assumes that given the hoop stress $\sigma_{\theta\theta}$ at some small distance $r_0$ from the crack tip and the anisotropic tensile strength $T_{\theta\theta}$ normal to the plane oriented at an angle $\theta$ to the fiber axis, the crack will grow along the plane on which the ratio $R_0(r_0, \theta)$, as defined below, is a maximum. Thus, the condition for cracking direction can be expressed as

$$R_0(r_0,\theta) = \frac{\sigma_{\theta\theta}(r_0,\theta)}{T_{\theta\theta}}, \quad \left[\frac{\partial R_0}{\partial\theta}\right]_{\theta_0} = 0, \quad \left[\frac{\partial^2 R_0}{\partial\theta^2}\right]_{\theta_0} < 0, \tag{17}$$

where $T_{\theta\theta}$ is dependent on the orientation of the angle $\theta$.

We follow the method of Buczek and Herakovich [9] for defining $T_{\theta\theta}$, which must satisfy the following conditions: (1) for an isotropic material, $T_{\theta\theta}$ must not depend on $\theta$; (2) for crack growth parallel to material fibers, $T_{\theta\theta}$ must equal the transverse tensile strength, $Y_T$; (3) for crack growth perpendicular to the fibers, $T_{\theta\theta}$ must equal the longitudinal tensile strength, $X_T$. Then $T_{\theta\theta}$ can be expressed as:

$$T_{\theta\theta} = X_T \sin^2\theta + Y_T \cos^2\theta \tag{18}$$

The hoop stress $\sigma_{\theta\theta}$ in the polar coordinate system is given by

$$\sigma_{\theta\theta} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta \tag{19}$$

Substituting the cartesian stress components of Equation (16) into the above expression, the hoop stress including the second order term can be obtained as

$$\begin{aligned}\sigma_{\theta\theta} = {} & \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}}\mathrm{Re}\left[\frac{1}{s_1 - s_2}\left(s_1\varphi_2^{3/2} - s_2\varphi_1^{3/2}\right)\right] + \frac{K_{\mathrm{II}}}{\sqrt{2\pi r}}\mathrm{Re}\left[\frac{1}{s_1 - s_2}\left(\varphi_2^{3/2} - \varphi_1^{3/2}\right)\right] \\ & + \sigma^{\infty}\mathrm{Re}[(\cos\alpha + s_1\sin\alpha)(\cos\alpha + s_2\sin\alpha)]\sin^2\theta\end{aligned} \tag{20}$$

where $\varphi_j = \cos\theta + s_j\sin\theta$ $(j = 1, 2)$. Thus, substituting the hoop stress of Equation (20) and the anisotropic tensile strength of Equation (18) into Equation (17), the normal stress ratio can be obtained as

$$R_0 = \frac{\begin{aligned}&\frac{K_{\mathrm{I}}}{\sqrt{2\pi r_0}}\mathrm{Re}\left[\frac{1}{s_1 - s_2}\left(s_1\varphi_2^{3/2} - s_2\varphi_1^{3/2}\right)\right] + \frac{K_{\mathrm{II}}}{\sqrt{2\pi r_0}}\mathrm{Re}\left[\frac{1}{s_1 - s_2}\left(\varphi_2^{3/2} - \varphi_1^{3/2}\right)\right] \\ &\quad + \sigma^{\infty}\mathrm{Re}[(\cos\alpha + s_1\sin\alpha)(\cos\alpha + s_2\sin\alpha)]\sin^2\theta\end{aligned}}{X_T\left[\sin^2\theta + (Y_T/X_T)\cos^2\theta\right]} \tag{21}$$

## RESULTS AND DISCUSSION

In order to investigate the effect of second-order stress term, we analyze the distribution of hoop stress at the crack tip and predict the initial crack extension angle for an uniaxially loaded sheet with an inclined crack as shown in Figure 1. The direction of crack extension is measured from the direction parallel to the $x$-axis passing through the point of crack initiation. It is assumed that the crack is aligned with direction $x$, which is parallel to the fiber orientation. As noted previously, the direction of crack extension is determined on the basis of maximum value of the normal stress ratio, $R_0$. We chose $0.01 \leq r_o/a \leq 0.05$. The reason is as follows. Eftis and Subramonian [10] compared analytical results for stresses with experimental data, and found that $r_0/a = 0.01$ is a reasonable value for isotropic materials under uniaxial load. Hence we used the same order of magnitude of distance in the present analysis.

## Distribution of Hoop Stress

The distribution of the hoop stress near the crack tip for the three cases of the crack angle, $\alpha$ equal to 15, 45 and 75° is analyzed. Figures 2(a) through 2(c) show the variation of normalized hoop stress, $\sigma_{\theta\theta}/\sigma^{\infty}$ with polar angle, $\theta$ for the case of $\alpha_0 = 1.2$ and $\beta_0 = 1.0$. The curves were obtained with the ratio, $r_0/a = 0.01$.

Figure 2(a) shows the distribution of hoop stress for the case of $\alpha = 15°$. The solid line in the figure is obtained using singular expression only for stresses near crack tip, and the dotted line is obtained using second-order term of the series expansion for the stresses. Both results agree well with each other for $\theta = 0$, but they start to differ when $\theta$ is non-zero, so it can be concluded that if the direction of crack extension deviates from the crack line, the effect of the second-order term may increase.

Figure 2(b) shows the hoop stress for the case of $\alpha = 45°$. The results obtained with singular expression agree well with the one obtained with second-order term all around $\theta$. There is little effect of the second-order term. It is therefore possible that the distribution of the hoop stress near crack tip can be accurately expressed using the singular term only.

The results for the case of $\alpha = 75°$ are given in Figure 2(c). In the figure, we note that the difference between the two results starts to appear again, when crack angles are away from 45°. The results are similar to that of the case of $\alpha = 15°$.

Based on these results, it is clear that the distribution of the hoop stress becomes dependent on crack angle. In particular, it is necessary to consider the subsequent term on the series expansion for crack tip stresses when crack angle deviates from 45°, which is changed by anisotropic material properties. Therefore, accurate calculation of hoop stresses may be required to predict crack extension angle, which is based on crack tip stresses.

## Prediction of Crack Extension Direction

The direction of initial crack extension for various values of tensile strength ratio $X_T/Y_T$ with $\alpha_0 = 1.2$ and $\beta_0 = 1.0$ was calculated. Figure 3 shows the effects of tensile strength ratio and second-order term on the direction of crack extension. The solid line in the figure is obtained using singular expression only for stresses near crack tip, and the dashed and dotted lines are obtained using second-order term for $r_0/a = 0.01$ and 0.05, respectively. As shown in the figure, the tensile strength ratio produces markedly different results for the predicted crack extension angle. For all crack inclination angles, the direction of crack extension approaches zero as the tensile strength ratio increases, and crack extension therefore occurs on the plane of the original crack. On the other hand, the

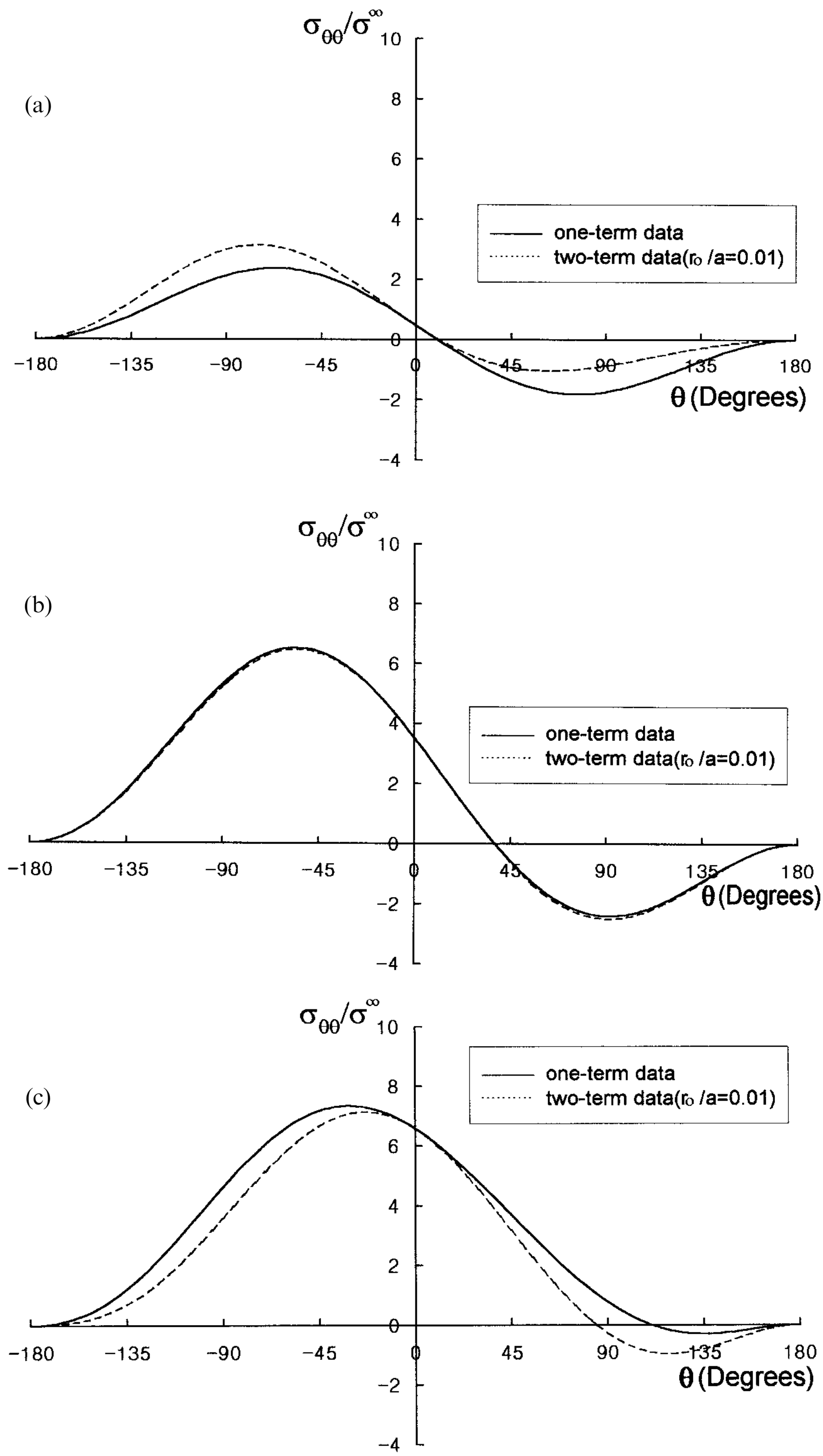


***Figure 2.*** *(a) Variation of $\sigma_{\theta\theta}$ for $\alpha = 15°$. (b) Variation of $\sigma_{\theta\theta}$ for $\alpha = 45°$. (c) Variation of $\sigma_{\theta\theta}$ for $\alpha = 75°$.*

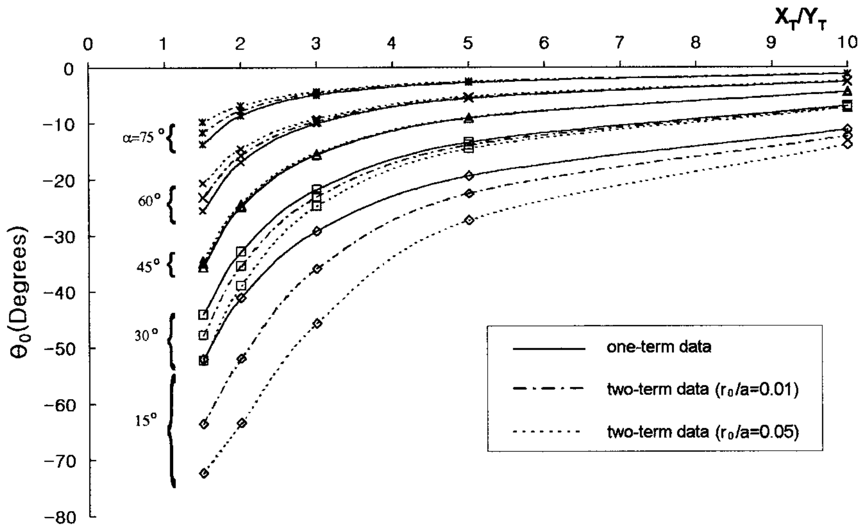


***Figure 3.*** *The effect of tensile strength ratio and second-order term on the direction of crack extension in anisotropic solid.*

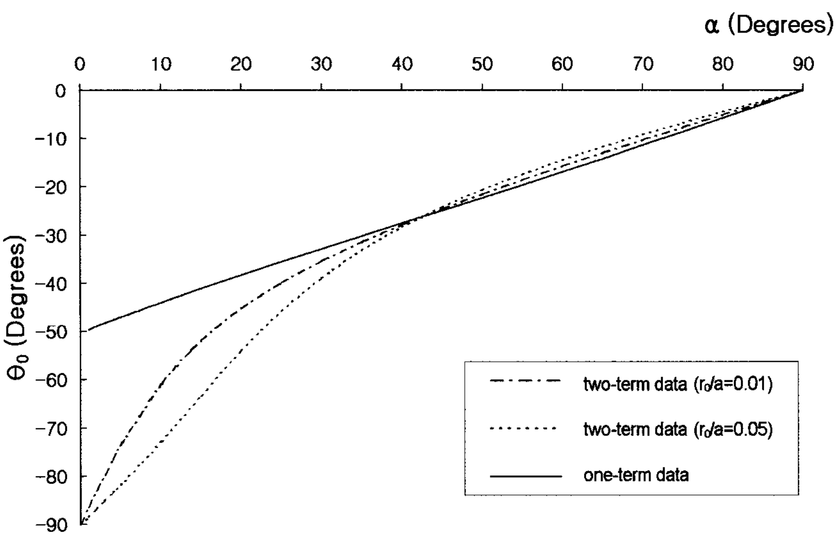


***Figure 4.*** *Direction of crack extension for an inclined crack in anisotropic solid.*

effects of second term on the predicted direction of crack extension are particularly significant as the tensile strength ratio becomes small. This graph further indicates that the effects of the second term on the predicted direction of crack extension generally increase as crack inclination angle $\alpha$ decreases. Further, the crack extension angle also varies with the distance $r_0$, where stresses are computed.

In order to represent the effect of the second order stress term clearly, we calculated $\alpha - \theta_0$ curves for the case of $\alpha_0 = 1.2,\ \beta_0 = 1.0$ and $X_T/Y_T = 2.0$. The results are shown in Figure 4. The solid line in the figure is obtained using singular expression only for stresses near crack tip, and the dashed and dotted lines are obtained using second-order term for $r_0/a = 0.01$ and 0.05, respectively. As shown in the figure, the inclusion of second term produces markedly different results for the crack extension angle. The effect of second term on the predicted direction of crack extension generally increases as crack inclination angle decreases. In addition, there is a difference with the value of $r_0/a$. This is the situation wherein Mode II is very active. It is therefore important to choose appropriate value of $r_0/a$ for the accurate prediction of crack extension angle in anisotropic materials. The critical value of $r_0/a$ for a given material should be determined by performing experiments. In particular, it is impossible to determine the crack extension angle in a vertical crack with $\alpha = 0$ using singular expression only, because the maximum normal stress ratio does not appear in this case, but the direction of crack extension using second-order term was predicted to be 90°.

The predicted directions of initial crack extension are compared in Table 1 for various tensile strength ratios and two $r_0/a$ ratios.

The direction of initial crack extension for the variation of tensile strength ratio, $X_T/Y_T$ and elastic modulus ratio, $\alpha_0$ and with $\beta_0 = 1.0$ and $r_0/a = 0.01$ is calculated. Figure 5 shows the effects of tensile strength ratio and elastic modulus ratio on the direction of crack extension. The solid, dashed and dotted lines in the figure are obtained for the case of $\alpha_0 = 1.2$, 2.0 and 3.0, respectively. As shown in the figure, it can be seen that the predicted propagation direction becomes more dependent on the elastic modulus ratio as the tensile strength ratio has small values. For all crack inclination angles, the effect of elastic modulus ratio on the predicted propagation direction decreases gradually as the tensile strength ratio increases.

The comparison between predicted directions of initial crack extension for various tensile strength ratios and three elastic modulus ratios is given in Table 2.

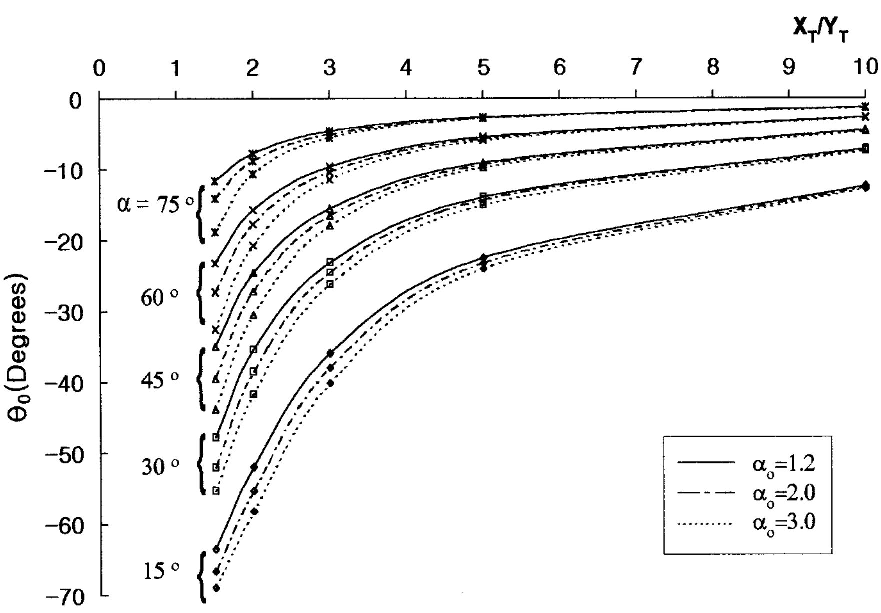


***Figure 5.*** *The effect of tensile strength ratio and elastic modulus ratio on the direction of crack extension in anisotropic solid.*

***Table 1. Comparison of the effects of tensile strength ratio and second-order term on the direction of crack extension in anisotropic solids ($\alpha_0 = 1.2$, $\beta_0 = 1.0$).***

| | Direction of Crack Extension, $\theta_0$ (Degrees) | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $X_T/Y_T$: | 1.5 | | | 2.0 | | | 3.0 | | | 5.0 | | | 10.0 | | |
| $r_0/a$: | one-term | 0.01 | 0.05 | one-term | 0.01 | 0.05 | one-term | 0.01 | 0.05 | one-term | 0.01 | 0.05 | one-term | 0.01 | 0.05 |
| $\alpha$ | | | | | | | | | | | | | | | |
| 0° | N/A* | ±90.0 | ±90.0 | N/A* | ±90.0 | ±90.0 | N/A* | ±90.0 | ±90.0 | N/A* | ±90.0 | ±90.0 | N/A* | ±90.0 | ±90.0 |
| 15° | −-52.0 | −63.5 | −72.2 | −41.1 | −51.9 | −63.3 | −29.2 | −35.9 | −45.6 | −19.4 | −22.5 | −27.3 | −11.2 | −12.3 | −13.9 |
| 30° | −44.1 | −47.8 | −52.3 | −32.9 | −35.4 | −38.9 | −21.8 | −23.1 | −24.8 | −13.4 | −13.9 | −14.5 | −6.9 | −7.1 | −7.3 |
| 45° | −35.6 | −35.0 | −34.3 | −24.9 | −24.6 | −24.2 | −15.6 | −15.5 | −15.3 | −9.0 | −9.0 | −8.9 | −4.4 | −4.4 | −4.4 |
| 60° | −25.6 | −23.2 | −20.6 | −16.9 | −15.7 | −14.5 | −10.0 | −9.6 | −9.1 | −5.5 | −5.4 | −5.2 | −2.6 | −2.6 | −2.5 |
| 75° | −13.7 | −11.6 | −9.7 | −8.6 | −7.7 | −6.8 | −4.9 | −4.6 | −4.3 | −2.7 | −2.6 | −2.5 | −1.2 | −1.2 | −1.2 |
| 90° | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |

N/A*: Not available.

**Table 2. Comparison of the effects of tensile strength ratio and elastic modulus ratio on the direction of crack extension in anisotropic solids ($X_T/Y_T = 2.0$, $\beta_0 = 1.0$).**

| | Direction of Crack Extension, $\theta_0$ (Degrees) | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $X_T/Y_T$: | 1.5 | | | 2.0 | | | 3.0 | | | 5.0 | | | 10.0 | | |
| $\alpha_0$: | 1.2 | 2.0 | 3.0 | 1.2 | 2.0 | 3.0 | 1.2 | 2.0 | 3.0 | 1.2 | 2.0 | 3.0 | 1.2 | 2.0 | 3.0 |
| $\alpha$ | | | | | | | | | | | | | | | |
| 15° | −63.5 | −66.6 | −68.9 | −51.9 | −55.3 | −58.2 | −35.9 | −37.9 | −40.1 | −22.5 | −23.2 | −24.0 | −12.3 | −12.5 | −12.7 |
| 30° | −47.8 | −52.0 | −55.3 | −35.4 | −38.5 | −41.7 | −23.1 | −24.5 | −26.2 | −13.9 | −14.3 | −15.0 | −7.1 | −7.2 | −7.4 |
| 45° | −35.0 | −39.5 | −43.8 | −24.6 | −27.2 | −30.5 | −15.5 | −16.5 | −17.9 | −8.9 | −9.3 | −9.7 | −4.4 | −4.4 | −4.6 |
| 60° | −23.2 | −27.3 | −32.5 | −15.7 | −17.7 | −20.7 | −9.6 | −10.3 | −11.4 | −5.4 | −5.6 | −5.9 | −2.6 | −2.6 | −2.7 |
| 75° | −11.6 | −14.1 | −18.8 | −7.7 | −8.8 | −10.6 | −4.6 | −5.0 | −5.5 | −2.6 | −2.7 | −2.8 | −1.2 | −1.2 | −1.3 |

## CONCLUSION

We have demonstrated the importance of retaining the second-order term in the series expansion of the local stresses for accurately predicting the crack propagation direction in an anisotropic plate with an inclined crack subjected to a tensile load. The analysis is based on the normal stress ratio, which is an empirical failure theory. The evaluation of the subsequent term of the series representation for the stresses in anisotropic materials and its effect on the predicted crack propagation direction are considered. It is concluded that the hoop stress near the crack tip and the crack extension angle are very much dependent on the tensile strength ratio and the second-order stress term. Results of this study have indicated that the prediction of crack extension using the subsequent term is important when the Mode II component in the crack is very significant and when the ratio of tensile strengths in the fiber and transverse directions deviates from unity.

## REFERENCES


1. Sih, G.C. and Liebowitz, H. (1968). In: Liebowitz, H. ed., *Fracture Mechanics*, Vol. II, pp. 108–130, Academic Press, New York.
2. Sih, G.C. (1981). In: Sih, G.C. ed., *Cracks in Composite Materials*, Martinus Nijthoff, Hague.
3. Yang, S. and Yuan, F.G. (2000). *International Journal of Fracture*, **101**: 291–319.
4. Yang, S. and Yuan, F.G. (2000). *International Journal of Solids and Structures*, **37**: 6635–6682.
5. Lim, W.-K, Choi, S.-Y. (2000). *Transaction of the Korean Society of Mechanical Engineers*, **24**: 69–78 (in Korean).
6. Lim, W.-K, Choi, S.-Y., Sankar, B.V. (2001). *Engineering Fracture Mechanics*, **68**: 403–416.
7. Lekhnitskii, S.G. (1968). *Anisotropic Plates*, Gordon and Breach Science Publishers, New York.
8. Savin, G.N. (1961). *Stress Concentration Around Holes*, Pergamon Press, Oxford.
9. Buczek, M.B. and Herakovich, C.T. (1985). *Journal of Composite Materials*, **19**: 544–553.
10. Eftis, J. and Subramonian, N. (1978). *Engineering Fracture Mechanics*, **10**: 43–67.