ADVANCED INTEGRATION BY PARTS

One of the more important ways used to evaluate integrals in elementary integral calculus is the integration by parts procedure. This method follows from the fact that the differential product of two functions u and u is given by the chain rule as-

d(uv)=udv+vdu

On integrating this equality term by term one obtains the integration by parts formula-

$$\int udv = uv - \int vdu$$

The internet is replete with ways to evaluate integrals using this technique. A very good article involving the evaluation of the simple integral $\int xexp(x)dx$, designed for high schoolers and first year science and math majors, is given by Nancy Pi on U Tube. The acrynym she includes for finding u and v at the end of her presentation is not necessary and probably leads to some confusion among students. We want here to work out some more complicated indefinite integrals usually not encountered in an introductory calculus course.

Let us begin with an evaluation of the logarithmic integral-

J(n) = $\int x^n ln(x)dx$

Where n is a positive integer. We make the substitutions-

u=lnX du=(1/x)dx v=x^(n+1)/(n+1) and dv=x^n dx

into the above integration by parts formula to get-

J(n)=ln(x) x^(n+1)/(n+1)-$\int \frac{x^n}{n+1} dx$= [x^(n+1)/(n+1)]ln(x)-x^(n+1)/(n+1)^2

With this information we can write-

J(2)-(x^3/3)ln(x)-x^3/9 and J(3)=(x^4/4)ln(x)-x^4/16

Generalizing we find the closed form analytic solution-

J(n)=$\int x^n \ln(x)\, dx = \frac{x^{n+1}}{n+1}[\ln(x) - \frac{1}{(n+1)}]$

Consider next an exponential integral which, when having the limits x=0 to x=infinity, is just the Laplace transform of the power x^n with s set to one. That is, we have-

$$\int_{x=0}^{\infty} x^n \exp(-x)\, dx = \Gamma(n+1)$$

Let us use integration by parts to evaluate the indefinite version of this integral, namely,-

J(n)=$\int (x^n) exp(-x)dx$

Here choose u=x^n and dv=exp(-x)dx. This yields du=nx^(n-1)dx and v=-exp(-x). Substitution into the above integration by parts formula, we find-

$$J(n)=\int (x^n)\, exp(-x)dx = -(x^n)exp(-x) + n\int x^{(n-1)} \exp(-x)\, dx$$

We could next apply integration by parts to the integral on the right and then continue the procedure until the power of x is down to zero. There is, however, a much faster way to get our answer. We can write down –

J(1)=-exp(-x)(1+x)

J(2)=-exp(-x)(x^2+2x+2)

From it we conclude that-

J(n)=-exp(-x) (x^n+nx^(n-1)+n(n-1)x^(n-2)+…+n!)

If we were to put the Laplace transform limits on this integral one would recover n!=Γ(n+1) ,

Note in general there is an extra constant added to the J(n) result. This is realized but not added here.

As a third indefinite integral consider the sinusoidal integral-

$$J(n)=\int [\sin(x)]^n dx$$

Here we have u=sin(x)^(n-1) du=(n-1)sin(x)^(n-2)cos(x)dx, dv=sin(x)dx and v= -cos(x) .Substitution into the above integration by parts formula, produces, after a little manipulation, the result-

$$J(n)= \int \sin(x)^n\, dx = -\frac{\cos(x)\sin(x)^{n-1}}{n} + (n-1)/n) \int \sin(\mathrm{x})\text{^}(\mathrm{n}-2)$$

We could go on and repeat the integration by parts to the integral on the right and so on. We don't do this here but note from the present result that-

$$J(2)=\int \sin(x)^2\, dx = \left(\frac{1}{2}\right)[-\sin(x)\cos(x)\ + x\,]$$

From this follows that-

J(4)= (1/8)[-2cos(x)sin(x)^3-3cos(x)sin(x)+3x]

I leave it to the reader to verify this.

We point out that the chain rule in calculus can be extended to three variables u, v, and w. This form reads-

d(uvw)= uvdw+uwdv+vwdu

On integrating this produces the result-

$$\int uvdw = uvw - \int vwdu - \int uwdv$$

Although not encountered in elementary calculus, this extended integration by parts formula works great for cases where there are three function involved like, for example,-

J=$\int x\ sin(x)^2 dx$

Here we have u=x, v=sin(x) and w=-cos(x). Plugging into the above three function by parts expansion we find-

J=-xsin(x)cos(x)+$\int \sin(x)\cos(x)\,dx + \int xcos(x)\text{^}2dx$

=-xsin(x)cos(x)-cos(x)^2/2+x^2/2-J

This leaves us with-

J=$\int xsin(x)^2 dx =$(1/4)[-2xsin(x)cos(x)-cos(x)^2+x^2]

If we place limits on x from x=0 to x=π/2, we get-

$$\int_{x=0}^{\pi/2} x\ sin(x)^2 dx = (\pi^2+4)/16 = 0.866850..$$

The above discussion has shown that integration by parts and its extension to integrals involving the product of more than two functions is an effective way to obtain solutions to both indefinite and definite integrals. By combining variable substitution together with integration by parts is a very effective way to evaluate integrals analytically. One can always check the results for definite integrals by a numerical evaluation.

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