EXPRESSNG NUMBERS IN DIFFERENT BASES

INTRODUCTION:

The two most frequently used number systems employ the decimal base b=10 and the binary base b=2.To express a decimal number such as N=237 in binary one first writes things out as-

$$237=2^7+2^6+2^5+2^3+2^2+2^0=128+64+32+8+4+1$$

$$\rightarrow 11101101$$

One records a 1 if the power of 2 exists, otherwise a 0. Note that binary representations involve only two symbols 0 and 1. This makes this base b=2 system ideal for electronic computers where an on or off are represented by 1 or 0, respectively. For every day numerical calculations however one uses the b=10 base with the ten symbols used being 0,1,2,3,4,5,6,7,8,9. This decimal system allows shorter representations of numbers at the expense of requiring more symbols. We want in this note to discuss the representation of any number N in terms of a specified base b.

BINARY NUMBER SYSTEM:

We begin by looking at the binary representations for the numbers 0 through 9. We have-

$$1=2^0,\ 2=2^1,\ 3=2^1+2^0,\ 4=2^2,\ 5=2^2+2^0,\ 6=2^2+2^1,$$
$$7=2^2+2^1+2^0,\ 8=2^3,\ 9=2^3+2^0$$

Reading off the powers of two we get the conversion table-

decimal	1	2	3	4	5	6	7	8	9
binary	1	10	11	100	101	110	111	1000	1001

The basic operations in binary follow the same format as for the decimal system except that there are now only two distinct symbols 0 and 1 involved. The basic rules for addition are-

1+1=10 , 1+0=0 , 0+1=0 , and 0+0=0

Thus-

$$\begin{array}{r} 1101 \\ +\,1010 \\ \hline 10111 \end{array} \qquad \text{and} \qquad \begin{array}{r} 10100 \\ +\,11101 \\ \hline 110001 \end{array}$$

To convert between decimal and binary and visa-versa one can use the MAPLE computer commands-

convert(decimal number, binary, decimal)

and-

convert(binary number, decimal, binary)

Thus decimal N=3456 is equivalent to 110110000000 in binary. The ratio of the number of binary elements to the number of equivalent decimal elements is R=12/4=3. If one goes to larger numbers we find, for example,-

N=679414231893567 going to
=10011010011110110001110011111100101000011000111111

in binary. The element ratio is here R=50/15=3.3333. If one lets N go to infinity the ratio becomes R=log(10)/log(2)=3.321928.. .

To multiply two binary base numbers one has the basic laws-

0 x 0 = 0, 0 x 1 = 0, 1 x 0 = 0, 1 x 1 = 1 , and 1 x 1 x 1 = 11.

Thus 101 x 1101= 1000001. In decimal this is equivalent to 5 x 13 = 65.
Note to double a binary number one just adds a zero on the right. Thus 110→6 produces1100→12 and 11000→24.

TERNARY NUMBER SYSTEM:

Another number system often used is the Ternary (base= 3). This system uses three distinct symbols 0,1, and 2. To find the first ten forms correspondingto the digital equivalents of 0 through 9 we can use the MAPLE command-

convert(decimal form, base,3)

This produces the table-

decimal	1	2	3	4	5	6	7	8	9
ternary	1	2	10	11	12	20	21	22	100

From this table we can infer that $3^n \rightarrow 1$ followed by n zeros. So $3^4=81$ is equivalent to 10000 in ternary. Also $(3^p)-1 \rightarrow$ all 2s p times. So 80 in decimal equals 2222 in ternary. The ratio R between ternary and decimal elements equals log(10)/log(3)=2.0959.. as N approaches infinity. Also 3^p+1 in decimal becomes 1 (p-1) zeros followed by 1. So 82 decimal equals 10001 in ternary. Addition in ternary follows the laws–

$$1+1=2 \quad \text{and} \quad 1+2=10$$

Thus 211 +111=1022 and 1211+0102=2020. Multiplication in ternary follows the laws-

$$1x1=1,\ 1x2=2,\ 2x2=11$$

Thus we have in ternary that-

$$1001 \text{ x } 111=111111.$$

In decimal this is equivalent to 28 x 13=364.

BASE b NUMBER SYSTEM:

In addition to decimal, binary, and ternary Number Systems, there are numerous other number systems possible. Each of these has certain advantages or disadvantages over others. To keep the number of required symbols small one chooses a system where the base is less than b=10. Binary is ideal for this possibility. A larger number of symbols are required when b is large, say greater than b=10. The hexadecimal Number System (b=16) is such an example with a large number of symbols but rather short expressions for the length of large N.

Here are the names of the nine best known Number Systems-

Base b	Name
2	binary
3	ternary
5	quinary
6	senary
8	octal
10	decimal
16	hexadecimal
20	vigesimal
60	sexagesimal

It is not certain how these Number Systems first came into being. However, it is likely that the decimal (b=10) approach to number manipulation came from the fact that man first counted numbers by using his ten fingers. Perhaps the Maya in the Yucatan had a base twenty (b=20) counting system based upon also including their toes. The base sixty (b=60) of the Babylonians was probably connected with astronomical observations of year length and moon period.

To go from a decimal system to a base b system we use the Maple computer command-

convert(decimal number, base, b);

It will always produce the value 10 when the b is expressed in decimal format. So convert(5,base,5)=10. Also we always have convert(b^2,base,5)=100. In addition one finds that convert(8^3,base,8)=1000.

A thorough knowledge of the powers of bases 1 through 12 allows one to quickly express N in any chosen base b=1 through b=12. It also allows one to make rapid mental calculations involving products of numbers. For instance, if you are asked to multiply N= 256 x 196, in your mind you would do the following-

N=16^2(14^2)=2^10(49)=1024x50-1024=51200-1024=50176

No hand calculator or pencil and paper needed!

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