GENERATING ACCURATE VALUES FOR THE TANGENT FUNCTION

Several years ago we came up with a new method to accurately approximate the tangent function –

$$\tan(x)=\int_{t=0}^{x}\frac{dt}{(\cos t)^2}=\mathrm{x}+\frac{1}{3}\mathrm{x}^3+\frac{2}{15}\mathrm{x}^5+\frac{17}{315}\mathrm{x}^7+\frac{62}{2835}\mathrm{x}^9+\mathrm{O}(\mathrm{x}^{11}),$$

We want here to summarize the method and add a few new observations on the technique.

Historically the tangent function tan(x) was the first of the trigonometric functions to be used extensively in land and elevation measurements. The ancient Egyptians employed the seked as a tangent measurement in their pyramid constructions and the more modern concept of slope of a curve in calculus or the grade of a roadbed in civil engineering are also connected directly with the concept of a tangent. In terms of a right triangle, the tangent represents the ratio of the length of the far side of the triangle to the length of the base. Some of the more important identities involving the tangent are -

$$\tan(x+y)=\frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}\quad,\quad 1+\tan(x)^2=\frac{1}{\cos(x)^2}\quad \mathit{and}\quad \tan(\frac{1}{2}x)=\sqrt{\frac{1-\cos(x)}{1+\cos(x)}}$$

The function has the exact values-

$$\tan(0)=0,\ \tan(\frac{\pi}{8})=\sqrt{2}-1,\ \tan(\frac{\pi}{6})=\frac{1}{\sqrt{3}},\ \tan(\frac{\pi}{4})=1,\ \tan(\frac{\pi}{3})=\sqrt{3}\ \mathit{and}\ \tan(\frac{\pi}{2})=\infty.$$

Also one has the odd symmetry property tan(-x)=-tan(x) and the periodic behavior tan(x)=tan(x±nπ). It is sufficient to only have accurate values of tan(x) within 0<x<1 since the remaining values can be deduced from these identities. Thus, for example since tan(π/8)=sqrt(2)-1=0.414213.. , we know at once that tan(3π/8)=[2+sqrt(2)]/sqrt(2)=2.414213…It also follows from the above double angle formula that one has the general relation-

$$\tan(\frac{\pi}{4}+\Delta x)=\tan(\frac{\pi}{4}-\Delta x)+\frac{4\tan(\Delta x)}{1-\tan(\Delta x)^2}$$

For arbitrary x the evaluation of tan(x) involves evaluating its infinite series as given above. This approach becomes impractical when multiple digit accuracies are required since the series converges rather slowly when x becomes larger and approaches x=π/2. In the later case it is of advantage to employ a new approximation method with which we came up with several years ago. The essence of the method is to use integrals involving Legendre polynomials and to note that-

$$I(n,a)=\int_{t=0}^{1}P_{2n}(t)\cos(at)dt=N(n,a)\sin(a)+M(n,a)\cos(a)$$

and

$$J(n,a)=\int_{t=0}^{1}P_{2n+1}(t)\sin(at)dt=K(n,a)\sin(a)+L(n,a)\cos(a)$$

Here N, M, K, and L are polynomials in n and a and the Ps are Legendre polynomials. Note that the subscripts on P are important in order to prevent additional terms appearing in the value of the integrals. When n gets large and the value of 'a' remains small, the integrals approach a value of zero. This allows one to introduce the tangent approximations-

$$\tan(a)\approx-\frac{M(n,a)}{N(n,a)}=C(n,a)\quad and\quad \tan(a)\approx-\frac{L(n,a)}{K(n,a)}=S(n,a)$$

Both of these approximations should be quite good when n>>1 and a<1. The ratios C and S can be readily evaluated either analytically or by computer. They produce the list-

$$C(1,a)=\frac{3a}{3-a^2}$$

$$S(1,a)=\frac{15a-a^3}{15-6a^2}$$

$$C(2,a)=\frac{105a-10a^3}{105-45a^2+a^4}$$

$$S(2,a)=\frac{945a-105a^3+a^5}{945-420a^2+15a^4}$$

$$C(3,a)=\frac{10395a-1260a^3+21a^5}{10395-4725a^2+210a^4-a^6}$$

$$S(3,a)=\frac{135135a-17325a^3+378a^5-a^7}{135135-62370a^2+3150a^4-28a^6}$$

$$C(4,a)=\frac{2027025a-270270a^3+6930a^5-36a^7}{2027025-945945a^2+51975a^4-630a^6+a^8}$$

$$S(4,a)=\frac{34459425a-4729725a^3+1351355a^5-990a^7+a^9}{34459425-16216200a^2+945945a^4-138600a^6+45a^8}$$

All the approximations are seen to go as 'a' when 'a' gets small enough. To check the accuracy of these approximations at a=1, we find that tan(1)= 1.55740772465490223…is approximated by-

$C(1,1) = 1.50...$

$S(1,1) = 1.555...$

$C(2,1) = 1.5573...$

$S(2,1) = 1.5574074...$

$C(3,1) = 1.557407722...$

$S(3,1) = 1.55740772464...$

$C(4,1) = 1.5574077246548...$

$S(4,1) = 1.5574077246549020...$

So the accuracy increases with increasing n as expected. The 15 digit accuracy of tan(1) given by S(4,1) is impressive considering that here 'a' lies at the limit of the typical range $0<a<1$ which we deal with below. The points at which tan(x) becomes infinite can be estimated by finding the value for which the denominators in the above approximations vanish. Thus S(2,a) predicts an infinity near a= 1.5708… which agrees with $\pi/2$ to three significant digits.

To carry out highly accurate estimates for the values of tan(x), we will make use of the double-angle formula-

$$\tan(x) = \frac{[\tan(x_0) + \tan(x - x_0)]}{[1 - \tan(x_0)\tan(x - x_0)]}$$

Here x_0 has a known value of $\tan(x_0)$ and lies close to x so that $x-x_o<<1$. Let us demonstrate the evaluation procedure for the case of $x=40°=2\pi/9$ radians and $x_o=\pi/4$. Thus $x-x_o=-\pi/36=-0.08726646…$. We find, using the C(3,a) approximation, that-

$$\tan(x) = \frac{[1 - \tan(\frac{\pi}{36})]}{[1 + \tan(\frac{\pi}{36})]} \approx \frac{[1 - C(3,\frac{\pi}{36})]}{[1 + C(3,\frac{\pi}{36})]}$$

Evaluating the last quotient to 30 significant digits produces the approximation-

$$\tan(2\pi/9) \approx 0.839099631177280011763147935518$$

This compares with the standard tabulated result(see Abramowitz and Stegun, Handbook of Mathematical Functions, pg199) where the tangent of 40 degrees is given as-

$$\tan(2\pi/9)=0.839099631177280$$

and the eighty digit accurate result given by our MAPLE computer program reads-

$\tan(2\pi/9)=$
0.83909963117728001176312729812318136468743428301234653324410203923251832805503452

Our C(3,a) approximation is seen to yield a value for tan(40deg) accurate to 22 decimal places. Even more accurate results follow by letting n>3.

This result shows that it is possible to quickly generate tables for the tangent function to any order of accuracy by just making n large enough and keeping 'a' small. Generating such extensive tables would today be a waste of time considering that any PC with a built in mathematics program can easily generate values for trigonometric functions to very high order of accuracy. It was a different story during WWII when mathematicians and their aides spent thousand of hours generating 15 digit accurate trigonometric tables using only mechanical calculating machines.

Other trigonometric function values follow directly from the present tangent approximation by use of the identities-

$$\cos(x)=\frac{1}{\sqrt{1+\tan(x)^2}}\approx\frac{1}{\sqrt{1+(CorS)^2}}\quad and \quad \sin(x)=\frac{\tan(x)}{\sqrt{1+\tan(x)^2}}\approx\frac{(CorS)}{\sqrt{1+(CorS)^2}}$$

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