EVALUATION OF THE COMPLETE ELLIPTIC INTEGRALS BY THE AGM METHOD

The complete elliptic integrals of the first and second kind are defined by the integrals-

$$K(m) = \int_{t=0}^{1} \frac{dt}{\sqrt{(1-t^2)(1-m^2t^2)}} \quad and \;\; E(m) = \int_{t=0}^{1} \sqrt{\frac{(1-m^2t^2}{(1-t^2}}\, dt$$

, respectively. Here o<m^2<1. We can convert these definite integrals to the semi-infinite range-

$$K(m) = \int_{x=0}^{\infty} \frac{dx}{\sqrt{(1+x^2)(1+(1-m^2)x^2)}} \quad \text{and} \quad E(m) = \int_{x=0}^{\infty} \sqrt{\frac{1+(1-m^2)x^2}{(1+x^2}}\, \frac{dx}{(1+x^2)}$$

by use of the transformation t=x/sqrt(1+x^2). We can also rewrite this last equation as-

$$K(m) = \frac{1}{\sqrt{1-m^2}} \int_{x=0}^{\infty} \frac{dx}{\sqrt{(a_0b_0 + x^2)(\frac{(a_0+b_0)^2}{4} + x^2)}}$$

where a_0b_0=1 and (a_0+b_0)/2=1/sqrt(1-m^2). One recognizes that the constant terms in the radical are just the geometric mean and the arithmetic mean of the numbers a_0 and b_0. If one now carries out the iterations-

$$a_{n+1} = \frac{(a_n+b_n}{2} \;\; \text{and} \;\; b_{n+1} = \sqrt{a_nb_n}$$

both a_{n+1} and b_{n+1} are found to approach the same limit M to a high order of accuracy after just a few iterations. The result, as first noted by Gauss, is that K(m) can be re-written as-

$$K(m) = \frac{1}{\sqrt{1-m^2}} \int_{x=0}^{\infty} \frac{dx}{M^2+x^2} = \frac{\pi}{2M\sqrt{1-m^2}}$$

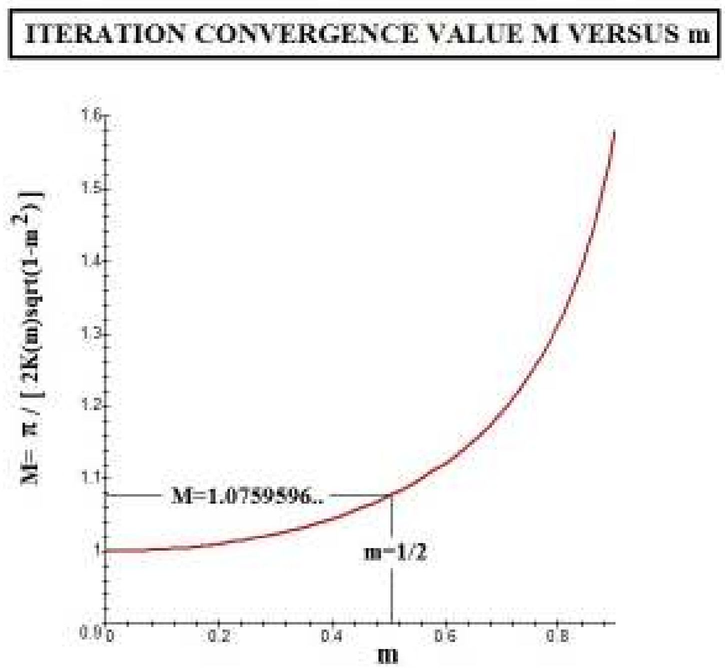
in closed form. The problem of determining the numerical value of K(m) is thus reduced to a simple iteration problem of finding M to any desired accuracy.

The first few iterations go as follows-

$$a_0 = \Delta + \sqrt{\Delta^2 - 1} \quad \text{and} \quad b_0 = \Delta - \sqrt{\Delta^2 - 1}$$

with $a_1 = \Delta$, $b_1 = 1$, $a_2 = \frac{(1+\Delta)}{2}$, $b_2 = \sqrt{\Delta}$, $a_3 = \frac{1+\Delta+2\sqrt{\Delta}}{4}$ $and\ b_3 = \sqrt{\frac{(1+\Delta)\sqrt{\Delta}}{2}}$

Here we have set Δ=1/sqrt(1-m^2). It is easy to carry out these iterations to higher order n via computer. For m=1/2 , corresponding to Δ=2/sqrt(3), one has at the third iteration a_3=1.075960.. and b_3=1.075959.. so that the approach to M =π/(K(1/2)sqrt(3)= 1.07595965147.. is already quite evident. Since most mathematics programs such as MAPLE and MATHEMATICA already have built in programs for elliptic integrals, we can work things backwards to the following graph of M =π/[2K(m)sqrt(1-m^2)] versus m-



One can use this graph to check the iteration convergence. For example at m=0.8 one should expect M to be very near 1.31. Note that M approaches M=1 at m=0 and M→∞ at m=1.

To evaluate E(m) we first need some extra definitions including K'(m^2)=K(1-m^2) and E'(m^2)=E(1-m^2) and the use the **Legendre Relation**(see Abramowitz and Stegun,"Handbook of Mathematical Functions")-

$$E(m^2)K'(m^2)+E'(m^2)K(m^2)-K(m^2)K'(m^2)=\pi/2$$

(Note that the m in our definition and most canned mathematics programs uses m^2 in the definition integral while Stegun and Abramowitz use m in their definition. Be aware of this difference.)

At m=1/sqrt(2) we have K(1/sqrt(2))=1.854074677 and-

$$E(1/sqrt(2)) \ = \frac{[K\left(\frac{1}{\sqrt{2}}\right)^2+\frac{\pi}{2}]}{[2K(\frac{1}{\sqrt{2}})}=\frac{1}{\sqrt{2}}\int_0^{\infty}\sqrt{\frac{2+x^2}{1+x^2}}\ \frac{dx}{(1+x^2)}=1.350643881$$

This result leads at once to a formula for evaluating π to any desired order of accuracy since it shows that-

$$\frac{\pi}{2}=K(\frac{1}{\sqrt{2}})\left[2E\left(\frac{1}{\sqrt{2}}\right)-K(\frac{1}{\sqrt{2}})\right]$$

To find E(m) for other values for m is a bit more complicated. One starts with the identity-

$$E(m)=(1-m^2)\left\{K(m)+m\frac{dK(m)}{dm}\right\}$$

which can be established directly by differentiation with respect to m. For m=1/sqrt(2) this produces an even more compact formula for π, namely-

$$\pi=\frac{1}{\sqrt{2}}\left\{\frac{dK(m)^2}{dm}\right\} \quad \text{at m=1/sqrt(2)}$$

or its equivalent –

$$\pi=\frac{M^2}{1-\frac{1}{M\sqrt{2}}\left[\frac{dM}{dm}\right]} \quad \text{at} \quad \text{m=1/sqrt(2)}$$

For numerical evaluations we replace dM/dn by its difference formula

$$\frac{dM}{dm}=\ \frac{lim}{\varepsilon\to 0}\left\{\frac{M\left(\frac{1}{\sqrt{2}}+\varepsilon\right)-M(\frac{1}{\sqrt{2}}-\varepsilon)}{2\varepsilon}\right\}$$

At the different value m=1/2, the E(m) function yields-

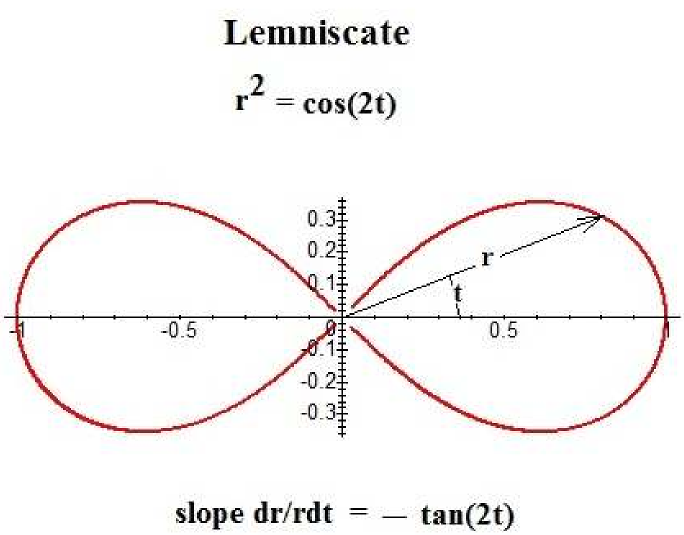
$$E(1/2)=1.467462209..=(3/4)\left\{1.68575035..+\left(\frac{1}{2}\right)541731848..\right\}$$

Since we know how to calculate K(m) to any order of accuracy by the AGM method, we can calculate its derivative and hence find the value of E(m) for any 0<m<1. In terms of M, m and Δ-1/sqrt(1-m^2) we find-

$$E(m)=\frac{\pi}{2}\left\{\frac{1}{M\Delta}+m\frac{d}{dm}\left(\frac{\Delta}{M}\right)\right\}$$

Note that the Legendre relation in conjunction with the AGM method is the basis for the latest calculations for π now known to nearly a trillion decimal places. The calculations involve m=1/sqrt(2) for which the Legendre relation assumes the particularly simple form given above.

Let us complete our discussion by looking at the perimeter of a lemniscate. It was this figure which initially led Gauss to the AGM technique. Its basic formula and graph follow-



Its perimeter is given by-

$$S=4\int_{t=0}^{\pi/4} r\,dt\sqrt{1+\tan(2t)^2}=4\int_{t=0}^{\pi/4}\frac{dt}{\sqrt{\cos(2t)}}=4\int_{t=0}^{\pi/4}\frac{dt}{\sqrt{1-2\sin(t)^2}}$$

Next, making the substitution x=sqrt(2) sin(t) we get-

$$S=4\int_{x=0}^{1}\frac{dx}{\sqrt{(1-x^2)(2-x^2)}}=\frac{4}{\sqrt{2}}K\left(\frac{1}{\sqrt{2}}\right)=5.2441151\ldots$$

Thus the perimeter of a lemniscate is expressed in terms of a complete Elliptic Integral of the First Kind and thus its solution is obtainable via math tables or the AGM method. To apply the AGM method one needs to first make one more coordinate transformation , namely, $x=U/\sqrt{U^2+1}$. This produces the following definite integral over the semi-infinite range-

$$S=4\int_{U=0}^{\infty}\frac{dU}{\sqrt{(1+U^2)(2+U^2)}}$$

Treating this integral by the AGM method we use Δ=2 since m=1/sqrt(2). Running through the iterations we have-

a_0=sqrt(2)+1, b_0=sqrt(2)-1, a_1=sqrt(2), b_1=1, a_2=(1+sqrt(2))/2, $b_2=(2)^{1/4}$.

Continuing , we find after seven iterations the result-

a_7 := 1.19814023473559220743992249228032387822721266321565155826363\

7495294640521414391567083588784

b_7 := 1.19814023473559220743992249228032387822721266321565155826363\

7495294640521414391567083588326

Thus we have a value for $M \approx a_7 \approx b_7$ good to 87 decimal places. This allows us to write that the perimeter of the Lemniscate as-

$$S=4\int_0^{\infty} \frac{dU}{M^2+U^2} dU = \frac{2\pi}{1.1981402347355922\ldots\ldots} =$$

5.24411510858423962092967917978223882736550990286324632563364340760158117414082850046058…

It is truly amazing that K.Gauss was able to come up with his AGM method nearly 213 years ago way before anyone even dreamed of electronic computers. I guess this is one of the reasons he is referred to as the "Prince of Mathematicians" (princeps mathematicorum). Note that the circumference of the ellipse $(x/a)^2+(y/b)^2=1$, by the same line of reasoning used for the lemniscate, is-

$$S=4bE\left(\sqrt{\frac{b^2-a^2}{b^2}}\right)$$

For the special case of a=1/sqrt(2) and b=1 we have S=4E(1/sqrt(2))=5.40257…

May 30, 2012