SOLVING COMPLETE ELLIPTIC INTEGRALS OF THE FIRST KIND BY THE AGM METHOD OF GAUSS

A very neat method to quickly evaluate elliptic integrals of the first kind is by use of the algebraic-geometric mean. The procedure works as follows. Starting with the definition-

$$K(m)=\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-m(\sin\theta)^2}}$$

we introduce the variables t= sinθ= u/√(1+u^2) to get the alternate forms-

$$K(m)=\int_0^1\frac{dt}{\sqrt{(1-t^2)(1-mt^2)}}=\int_0^\infty\frac{du}{\sqrt{(1+u^2)(1+(1-m)u^2)}}$$

Now , as first noted by Gauss , the algebraic-geometric mean (AGM) of a_0=1 and b_0=1/sqrt(1-m) will approach an identical value of M as one carries out the iterations-

$$a_{n+1} = (a_n+b_n)/2, \text{ and } b_{n+1} = \sqrt{(a_n b_n)}$$

Using the substitution 2v=u-(a_0b_0)/u one finds the last integral above can be rewritten and then integrated exactly as follows-

$$K(m)=\frac{1}{\sqrt{(1-m)}}\int_0^\infty\frac{dv}{\sqrt{[a_0b_0+v^2][((a_0+b_0)/2)^2+v^2]}}$$

$$=\frac{1}{\sqrt{(1-m)}}\int_0^\infty\frac{dv}{(M^2+v^2)}=\frac{\pi}{2M\sqrt{(1-m)}}$$

To demonstrate, consider the special case of K(0.5). Here we have a_0=1 and b_0=√2 and after four iterations we find the 19 place accurate result a_4=b_4=1.1981402347355922075. We thus have-

$$K(0.5)=\frac{\pi}{M\sqrt{2}}=1.8540746773013719184$$

The 15 place accurate math tables of Abramowitz and Stegun give the identical value K(0.5)=1.854074677301372.

Note that this AGM approach , which also works for complete elliptic integrals of the second kind, has found applications in recent years in the numerical determination of π to over 100 billion places with aid of supercomputers.