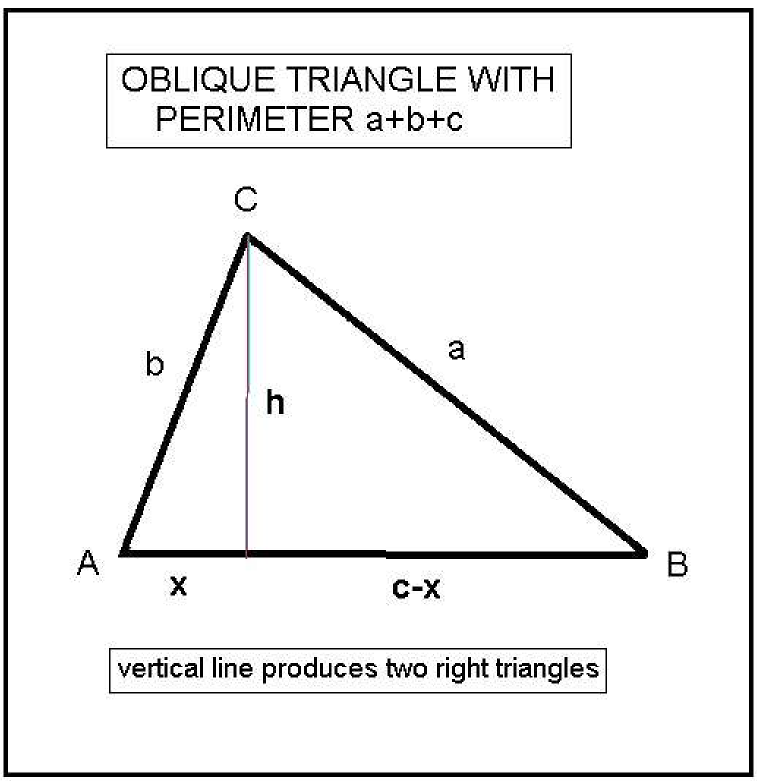
ALL ABOUT TRIANGLES

The triangle is the simplest of all two dimensional figures consisting of just three vertexes A, B, and C plus three sides a, b, and c. It is the historical basis for all trigonometric functions plus the Pythagorean Theorem, the Sine and Cosine Theorems, and the famous Hero's Formula. Here is a picture of such an oblique triangle-



We want in this note to discuss all properties of this triangle in a sometimes unconventional manner using the two right triangles shown. By applying the Pythagorean Theron, one has-

$$h^2+x^2=b^2 \qquad \text{and} \qquad h^2+(c-x)^2=a^2$$

Eliminating the interior vertical line length h, we get-

$$x = \frac{(c^2 - a^2 + b^2)}{2c}$$

Next solving for h we have-

h= [1/(2c)]sqrt[(2bc)^2-(c^2-a^2+b^2)^2]

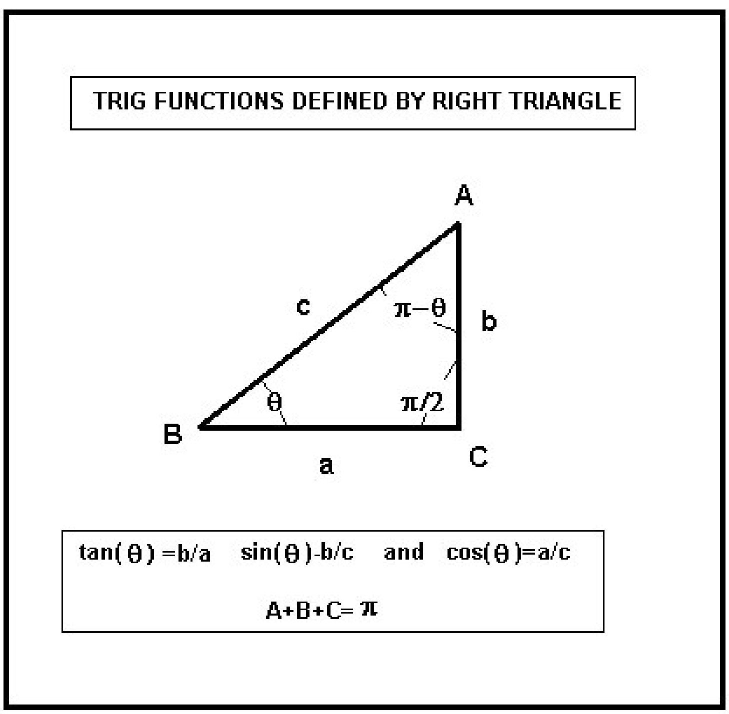
Now we know that the area of the entire oblique triangle will be hc/2. Hence we have-

Area of Oblique Triangle = (1/4) sqrt{(2bc)^2-{c^2-a^2+b^2)^2}

This is just the Heron Formula relating the area of a triangle to only its side-lengths without requiring any angle knowledge. Heron solved this problem some 2000 years ago using a much longer geometrical proof involving an inner circle. His result is expressed as a function of the semi-perimeter s=(a+b+c)/2. It reads-

Area of Oblique Triangle=sqrt[s(s-a)(s-b)(s-c)]anor an example take the equilateral triangle of side length unity. We get a=b=c=1 and an area sqrt(3)/4.

Next let us define the basic trigonometric functions using the following figure-



We have there a generic right triangle with all vertexes and side-lengths shown. The sum of the angles of this triangle is always π radians. We define the basic functions associated with this right triangle as-

tan(B)=b/a sin(B)=b/c cos(B)=a/c

Actually one requires only the tangent in the range $0< \theta<\pi/4$ to get the values for these three functions plus the less used sec, csec and, cot functions for any angle. This follows from the fact that-

sin(B)=tan(B)/sqrt[1+tan(B)^2] and cos(B)=1/sqrt[1+tan(B)^2]

We can also generate further triangle laws starting with the oblique triangle graph at the introduction to this triangle. Recall that the vertical line interior to the triangle has value h and we can read off from the graph that sin(B)=h/a and sin(A)=h/b. This produces, upon introducing the third angle contribution, the Law of Sines-

$$\frac{\sin(A)}{a}=\frac{\sin(B)}{b}=\frac{\sin(C)}{c}=d=\text{constant}$$

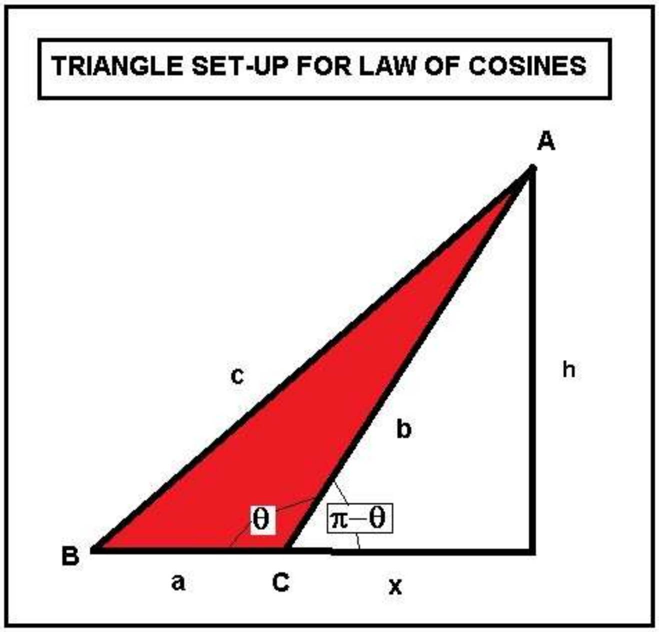
valid for any triangle. We can also write this result as-

$$\frac{a-b}{a+b}=\frac{\sin(A)-\sin(B)}{sin(A)+\sin(B)}=\frac{\tan[(A-B)/2]}{\tan[(A+B/2]}$$

This produces the Law of Tangents. Note that one can also write this Tangent Law as the ratio of the two Mollweide Equations-

(a+b)/c=cos[(A-B)/2]/cos[(A+B)/2] and (a-b)/c=sin[(A-B)/2]/sin[(A+B)/2]

Next we'll develop the Law of Cosines using the following two-triangle arrangement shown-



Here we can use the following three formulas to relate the cosine of the angle C to the side-lengths a,b, and c by eliminating h and x. We have-

$$\cos(\pi-\theta)=x/b$$

$$(a+x)^2+h^2=c^2$$

$$b^2+x^2=h^2$$

Eliminating first h and then x, one ends up with the Law of Cosines-

$$c^2=a^2+b^2-2ab\cos(\theta)$$

There are dozens of other formulas constituting trigonometry based on triangle geometry. Especially those involving half and double angle formulas. We will not go further into these at this time. In most cases such multiple angle formulas are straight forward to derive. Here is just one such example. We start with the sum of two angle formulas-

$$\sin(A+B)+\sin(A-B)=\sin(A)\cos(B)+\sin(B)\cos(A)+\sin(A)\cos(B)-\sin(B)\cos(A)=2\sin(A)\cos(B)$$

and then set A=B. This produces $\sin(2A)=2\sin(A)\cos(A)$, a well known result. One also has, by a related manipulation, that $\cos(2A)=2\cos(A)^2-1$ for any

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December 10, 2022
Gainesvile, Florida