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A NEW METHOD FOR OBTAINING HIGHLY ACCURATE APPROXIMATIONS TO THE ARCTAN FUNCTION

by

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INTRODUCTION:

Computing the arctangent is needed in many applied fields, for example, measuring distance on the ellipsoid for navigation, calculating phase in digital signal processing and determining angles in computer graphics. Each of these applications requires that the simple definition of the tangent on a right triangle be inverted. However, this is difficult as the arctangent exhibits a highly non-linear relationship with angle. Methods to calculate the arctangent are usually based upon summation of a Taylor series. This summation is inconvenient because the series converges slowly outside of a narrow range ($x<<1$). The conventional approach performs range reduction and attempts to shorten the series using minimax methods. Nevertheless, for high accuracy, many terms with lengthy coefficients must be summed (Hart et al, 1968 and Muller, 2005). Even in the last 10 years, there has been only marginal progress on computing arctan. Medina found a somewhat shorter series using Hermite polynomials (Medina, 2006). Other workers have sidestepped the convergence issue by deriving the arctangent from the arcsine (Markstein, 2005) or using brute force computer power to iterate to the solution using either the Cordic (Muller, 2005) or KDF9 methods (Findlay, 1964). Recent work for situations with thousands of measurements that need to be efficiently inverted has focused on low degree but inaccurate formulas covering the entire range [0,1] with just a few simple coefficients (Rajan et al, 2006). Some of these formulas try to avoid division which is a costly instruction in a CPU. In summary, previous methods have not been entirely satisfactory and the atan2 function as implemented in a computer library is slow and expensive.

It is the purpose of this note to introduce a new approach for finding highly accurate approximations for $\arctan(1/a)$ valid for the entire range $-\infty<a<+\infty$ suited for computer sub-routines. We also demonstrate a multipoint form of range reduction that enables us to calculate any arctangent starting from a nearby argument for which the arctangent is known.

NEW ARCTAN APPROXIMATION

Although very accurate values for arctan(x) expressed in fractions of π are known for x=0, ±1/sqrt(3), ±1, and ±sqrt(3) , other points in the range -∞<x<∞ rely on tables or computer sub-routines based on the basic series expansion

$$\arctan(x)=\sum_{k=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)}=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\ldots \qquad \textbf{(1)}$$

and the identity

$$\arctan(x)+\arctan(1/x)=\pi/2 \qquad (2)$$

Since arctan(x) is an odd function one has arctan(-x)= -arctan(x) and so one needs to only find values for x>0 to know the value of arctan(x) over the entire range [-∞,∞].
The problem with the series expansion shown in (1) is that it converges very slowly because it lacks a factorial term in the denominator of the series. This makes it difficult to obtain high precision for arbitrary values of x with standard computer sub-routines.

We show here a new approach to obtaining very accurate approximations to arctan (x) based on an integral evaluation method involving Legendre polynomials. The procedure, already outlined earlier (Kurzweg, 2009 and Kurzweg, 2011), is to start with the basic definition

$$F(a)=(1/a)\arctan(1/a)=\int_{t=0}^{1}\frac{dt}{t^2+a^2} \qquad (3)$$

and then look at the related integral

$$I(n,a)=\int_{t=0}^{1}\frac{P_{2n}(t)}{(t^2+a^2)}dt \qquad (4)$$

Here $P_{2n}(t)$ represents the even Legendre Polynomials which can be defined by the Rodrigues Formula as

$$P_{2n}(t)=\frac{1}{2^{2n}(2n)!}\frac{d^{2n}}{dt^{2n}}(t^2-1)^{2n} \qquad (5)$$

This differential expansion clearly shows that the $P_{2n}(t)$ polynomials are 2nd order even polynomials in t and that, when they are integrated over the half range 0<t<1, their integral always vanishes. They have an oscillatory character with exactly n zeros in [0,1]. It stands to reason that the integral I(n,a) should also approach zero when both a and n are large.

If one now expands the quotient in (4), one finds that

$$I(n,a) = \int_{t=0}^{1} Q(n,a,t)dt + M(n,a)F(a) \qquad (6)$$

Here Q(n,a,t) is the polynomial obtained by dividing $P_{2n}(t)$ by t^2+a^2 after omitting the term $M(n,a)/(t^2+a^2)$. We call M(n,a) the remainder term. Next, a simple integration, produces the identity

$$I(n,a) = N(n,a) + M(n,a)F(a) \qquad (7)$$

with N(n,a) and M(n,a) being even polynomials in 'a' for fixed n. If one sets I(n,a) to zero, the following new approximation for arctan is found

$$F(a) = (\frac{1}{a})\arctan(\frac{1}{a}) \approx \frac{-N(n,a)}{M(n,a)} \qquad (8)$$

This result shows that the function F(a) is approximated by the ratio of two even polynomials in 'a' for fixed n. The result is reminiscent of a Pade approximate (Mathews, 2003), but as we shall see is far easier to implement and is obtained without the usual complications of evaluating a large set of algebraic equations. The order of our approximation is determined by the value of n employed, with the larger values giving more accurate results.

Here are the resulting approximations for n=1 through n=4:

$$F(a) \approx \frac{3}{(1+3a^2)} \quad for \quad n=1$$

$$\approx \frac{5(11+21a^2)}{3(3+30a^2+35a^4)} \quad for \quad n=2$$

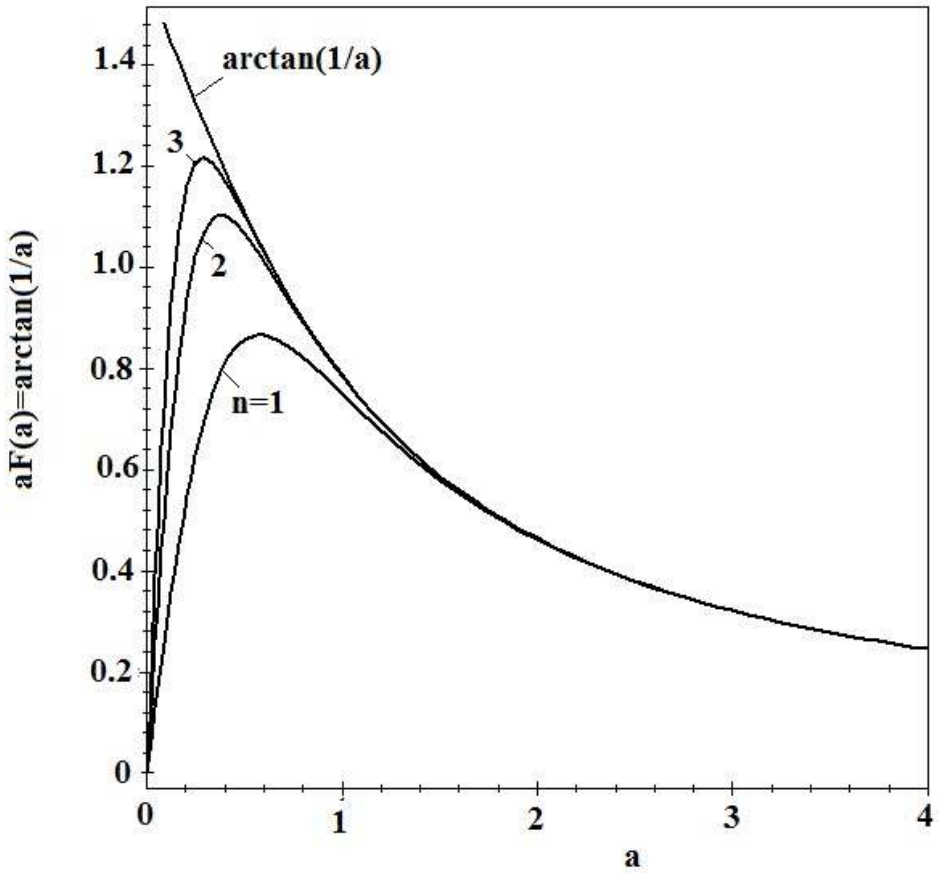
$$\approx \frac{7(33+170a^2+165a^4)}{5(5+105a^2+315a^4+231a^6)} \quad for \quad n=3$$

$$\approx \frac{(15159+147455a^2+345345a^4+225225a^6)}{35(35+1260a^2+6930a^4+12012a^6+6435a^8)} \quad for \quad n=4 \qquad (9)$$

The numerator in these approximations contains only even powers of ‘a’ up to the 2(n-1)th power while the denominator is a polynomial in ‘a’ containing only even powers up to 2n

To get a feel for the increasing accuracy of F(a) with increasing n we have made the following plot

Fig.1-Approximations for arctan(1/a) for n=1, 2, and 3



It is seen that accuracy improves with increasing n and a. The largest departures from the exact value of arctan(1/a) occur for a<1. At a=1 and a=5 we find the approximations to arctan(1)=π/4 and arctan(1/5) for n= 2, 4, and 8 are as shown in Table I.

TableI -Values for the 1F(1) and 5F(5) approximations with increasing n

n	1F(1)=arctan(1)	5F(5)=arctan(1/5)
2	0.78	0.19739555

4	0.78539	0.1973955598498807
8	0.78539816339	0.19739555984988075837004976519479
exact	0.7853981633974483096156	0.19739555984988075837004976519479029

Here the numbers have been terminated at the point where a departure from the exact value first occurs. The approximation for the case a=5 is excellent producing a 32 digit accuracy for n=8. The fact that the approximations are not very good when a<<1 comes from the fact that in the original integral I(n,a) the denominator t^2+a^2 varies appreciably across the interval 0<t<1 when a is small. This, however, does not preclude finding excellent estimates for arctan(1/a) when a<1 , if one makes use of formula (2).We will demonstrate this procedure in the next section.

We also wish to point out that formulas involving arctangents with large 'a' are nicely evaluated directly by the present approximations aF(a). Take the case of the interesting four term arctan formula for π found by us several decades ago. It reads

$$\pi = 48\arctan(\frac{1}{38}) + 80\arctan(\frac{1}{57}) + 28\arctan(\frac{1}{239}) + 96\arctan(\frac{1}{268}) \qquad (10)$$

Using an n=26 approximation, yields the 99 digit accurate result-

π=3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482534211 70

EVALUATION OF ARCTAN(1/a) FOR 0<a<1

As seen in Fig.1, our arctan approximations, involving the quotient of two polynomials, depart from the standard arctan(1/a) result in 0<a<1, with this departure becoming most noticeable when n becomes smaller. To get accurate approximations for a<1, we use equation (2) which is just a special case of the general arctan relation

$$\arctan(\frac{1}{a+\Delta a}) = \arctan(\frac{1}{a}) - \arctan\left[\frac{\Delta a}{a(a+\Delta a)+1}\right] \qquad (11)$$

which is valid for any Δa . In terms of 'a' , equation (2) reads

$$\arctan(\frac{1}{a}) = \frac{\pi}{2} - \arctan(a) \qquad (12)$$

Thus we can approximate all arctangents in 0<a<1 by the relation-

$$\arctan(1/a) \approx \frac{\pi}{2} + \frac{N(n,1/a)}{aM(n,1/a)} \qquad (13)$$

To demonstrate how this approximation is used, consider arctan(1/0.55) where a=11/20<1. Substituting into (13) and carrying out the calculation for n=8, produces the 17 digit accurate result-

$$\arctan(\frac{20}{11}) \approx 1.06795311586703579 \qquad (14)$$

Alternatively one could use (11) to first break things up into a three term identity and then evaluate two different Fs as shown here-

$$\arctan(\frac{20}{11}) = \frac{\pi}{4} + \arctan(\frac{1}{3}) - \arctan(\frac{2}{51}) \approx \frac{\pi}{4} + 3F(3) - \frac{51}{2}F(\frac{51}{2}) \qquad (15)$$

This time the n=8 approximation yields the 26 digit accurate result-

$$\arctan(\frac{20}{11}) = 1.0679531158670357919004396 \qquad (16)$$

The higher accuracy in this second case arises from the fact that one is dealing with larger values of a in the approximations. That is, 11/20<3<51/2.

We are now in a position to obtain highly accurate approximations to the arctangent functions over the entire range 0<a<∞. Arctangent values for negative values of 'a' are also recoverable since arctan(1/a) is an odd function. The process can be automated into a simple computer routine using a canned symbolic mathematics program such as MAPLE. We have used a simple MAPLE program to obtain approximations for arctan(1/a) for a=1/2, 1/4, 1/8, and 1/16 using n=10. The results are recorded in Table II

TableII -Approximations to arctan(1/a) for 1/a=2, 4, 8, and 16 using a MAPLE program generated from eq. (13) when n=10.

(1/a)	arctan(1/a) ≈π/2-(1/a)F(1/a)
2	1.10714871779409050301706 5
4	1.32581766366803246505923921042847563 1
8	1.44644133224813518419996684247588041652541450791 7
16	1.50837751679893927075734257865424632849231081189005371587994 4

Each result has been terminated at the point where a departure from the exact value first occurs. The accuracy is impressive. The accuracy of the approximations for fixed n is seen to increase with increasing values of 1/a. Note that in all our calculations thus far, the only inputs which have been used are the known values of arctan(∞)=π/2 and arctan(1) =π/4 .

Finally we show how to make highly accurate arctan approximations using multipoint range reduction. These approximations can use just n=3 or n=4 providing we start from a nearby argument where the arctangent has been previously calculated. A "ladder" procedure (see appendix) can generate arctangent values at a series of steps starting from a known point such as arctan(1)=π/4. If these steps are correctly spaced then we can control the accuracy of our approximation because we know in advance the minimum argument for F using equation 19.

For example, if we generate arctangents at the steps 0.95,0.85,0.75…0.05 then these steps can be used to define ranges (0.9,1),(0.8,0.9),(0.7,0.8)…(0,0.1). Then when, for example, we are presented with an argument of 0.9 we may use the second range to calculate arctan(1/0.9) from the previously calculated value of arctan(1/0.85) as follows-

$$\arctan(1/0.9) = \arctan(1/0.85) - \arctan[0.05/((0.85(0.85+0.05)+1)] \qquad (17)$$

Or the equivalent approximation

$$\arctan(\frac{1}{0.9}) \approx \arctan(\frac{1}{0.85}) - 35.3F(35.3) \qquad (18)$$

Using an argument of 0.1 and the arctangent at 0.05, and employing the generic form-

$$\arctan(\frac{1}{a+\Delta a}) \approx \arctan(\frac{1}{a}) - [\frac{a(a+\Delta a)+1}{\Delta a}]F[\frac{a(a+\Delta a)+1}{\Delta a}] \qquad (19)$$

the present approximation shows that the minimum F argument will be 20.1 using these 10 ranges. A computer sub-routine implementing equation (19) and using arctangents found at the 10 steps (0.05,0.15,0.25,…0.95) achieves accuracy better than 1.E-30 over the range [-infinity, infinity] using the n=4 approximation.

CONCLUDING REMARKS: We have found a new approximation technique for the arctan function which can obtain highly accurate values for arctan(1/a) over the entire range 0<a<∞. Some care must be taken when 0<a<1, but this difficulty can be readily overcome by using an arctan difference formula. The numerical accuracy is found to be remarkable especially when using the higher n approximations. A ninety nine digit accurate approximation for π based on a four term arctan formula has been obtained for n=26. Excellent approximations for arctan($1/2^k$) are also found via an easily employable

computer routine. Incorporation of the type of polynomial quotient approximation method introduced here should also find applications for built-in computer routines for other elementary functions.

APPENDIX: Procedure Ladder generates arctangents by iterating towards either a target argument or a series of equally spaced arguments (steps) in the range [0,1]. Starting from 1 with known arctangent π/4, it calculates a new result using equation (19) and the result from the previous loop and continues until it reaches each step and prints its result. The calculated values of the arctangent function at the steps can be used in code to calculate arctangents using equation (19) without iteration as shown in eq. (17).

```
Procedure Ladder(nsteps integer default 10, target number default 0.0)

--Beginning at arctan(1) = π/4
--Step towards various goals (0.95,0.85,0.75,..0.05) using equation (19):

--     arctan(1/(a+Δa) = arctan(1/a) – aa*F(aa) where aa = ((a(a+Δa)+1)/Δa

-- call these: result  = LH arctan   - RH term

   a := 1.;                                    -- start from "a" is 1
   LH_arctan = π/4;                            -- and set its known arctan
   stepsize = 1.0/nsteps;                      -- stepsize will be 0.1
   next_goal = a – stepsize*0.5;               -- we want to find arctan(0.95)

   if target <> 0.0 then
     next_goal = target;                       -- or for some specified argument
   end if;

   gap = 1/a - next_goal;                      -- gap to next goal
   Δa = orig_Δa = stepsize*0.1;                -- set small delta to add to "a"

   For ii in 1..40000 Loop                     -- loop towards the next goal
      aa :=( a*(a+Δa)+1.)/Δa;                  -- setup argument for RH term

   -- At loop 1, this is the computation:  a=1, Δa = 0.01 and aa = 201

   -- arctan(1/1.01) = arctan(1) - arctan(0.01/(1(1.01) +1) ≅ π/4 - 201F(201,n=4)

      result := LH_arctan – aa*F(aa,n=4);      -- F function is equation 9
      a = a + Δa;                              -- move 1/a towards next step

   -- Are we close enough to our goal?

      if (1.0/a) - next_goal <= gap and gap > 1.E-35 then

        gap = ABS(1.0/a - next_goal);          -- keep track of distance to goal
        Δa = min(gap,Δa);                      -- make Δa smaller as we get close
        best_result = result;                  -- save our best result before next goal

   -- When we get close enough, print best result

      else
         print next_goal, best_result          -- 0.95, .7597627548757.. on 1st iteration

         if target = 0.0 then                  -- if a series of steps desired then
           next_goal = next_goal - stepsize;   -- reset next goal to 0.85, 0.75 ..
         end if;
         gap = 1.;                             -- reset gap to find next step
         Δa = orig_Δa;                         -- and delta to a larger value
      end if;

      LH_arctan = result;                      -- reuse result in the next iteration

   End Loop;
   End Ladder;
```

Some results from procedure Ladder:

X	Arctangent	error	loop
.95	.75976275487577082892296119539998182 4	1.90E-38	38
.85	.70449406424221771665748034078199625 9	1.15E-38	110
.75	.64350110879328438680280922871732 2	4.30E-35	217
.65	.57637522059118368022757047839377005 1	-7.00E-39	375
.55	.50284321092786082733088202924527756 2	-2.54E-38	612
.45	.42285392613294071296648279098114198 0	-1.82E-38	985
.35	.33667481938672718139669863134176 64	1.48E-37	1623
.25	.24497866312686415417208248121127581 9	-4.28E-38	2900
.15	.14888994760949725058653039165586729 0	-3.95E-38	6460
.05	.04995839572194276141000628703484489	-4.62E-38	37678

REFERENCES:

J. F. Hart, E. W. Cheney, C. L. Lawson, H. J. Maehly, C. K. Mesztenyi, J. R. Rice, H. C. Thacher Jr., C. Witzgall, "Computer Approximations", Wiley, 1968, Lib. Cong. 67-23326.

J.H.Muller, "Elementary Functions, Algorithms and Implementation", Birkhauser Verlag, Boston, 2nd ed.2006.

H.A.Medina, "A Sequence of Polynomials for Approximating Inverse Tangents", Am.Math.Monthly,113,156-161, 2006 . Also found at- http://myweb.lmu.edu/hmedina/Papers/arctan.pdf

P.W.Markstein, "A Fast-start Method for Computing the Inverse Tangent", 17th IEEE Symposium on Computer Arithmetic, 2005.

S.Rajan, S.Wang, R.Inkol and A.Joyal,"Efficient Approximations for the Arctan function",IEEE Signal Process.Magazine ,111, May 2006.

W. Findlay, "The English Electric KDF9", http://www.findlayw.plus.com/KDF9/The%20English%20Electric%20KDF9.pdf 2011.

U.H.Kurzweg,"Evaluation of Arctan(1/a)", http://www.mae.ufl.edu/~uhk/EVAL-ARCTAN.pdf , Oct.2009.

U.H.Kurzweg,"Further Remarks on Finding Numerical Approximations for Arctan(1/a)", http://www.mae.ufl.edu/~uhk/MORE-EVALUATION-ARCTAN.pdf , May 2011.

J.H.Mathews,"Pade Approximations"
http://math.fullerton.edu/mathews/n2003/pade/PadeApproximationProof.pdf , 2003.