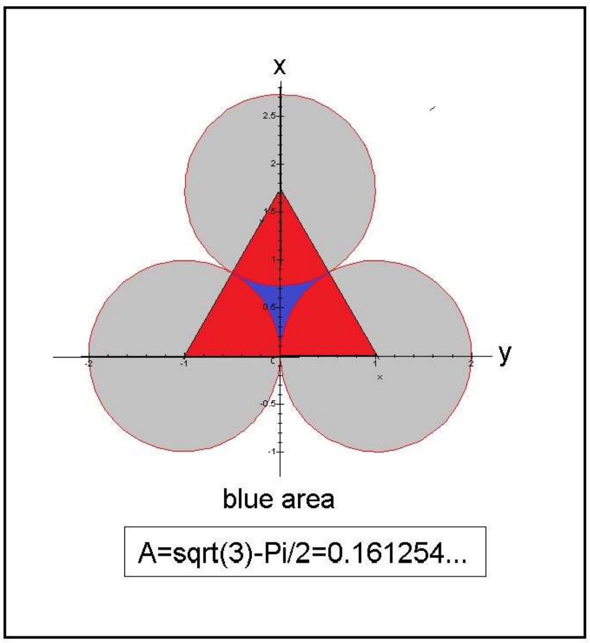
AREA DETERMINTION OF SUB-AREAS CREATED BY CIRCLES AND STR AIGHT LINES

If you look at the internet, there are hundreds of web pages discussing solutions of areas created by combining circles with straight lines. One of the more popular of such multiple discussions can be found by typing in Presh Talwalker , Mind Your Decisions into the Google search engine. It is our purpose here to generalize some of the math problems treated there, add a few new ones, and show that these problems can always be solved in closed form by the use of simple geometry and trigonometry familiar to must middle school students. No calculus is required, although it is often convenient to use calculus to verify a given answer.

Let us begin with the following trivial problem. One starts with three tightly packed circles of radius r=1 each as shown-



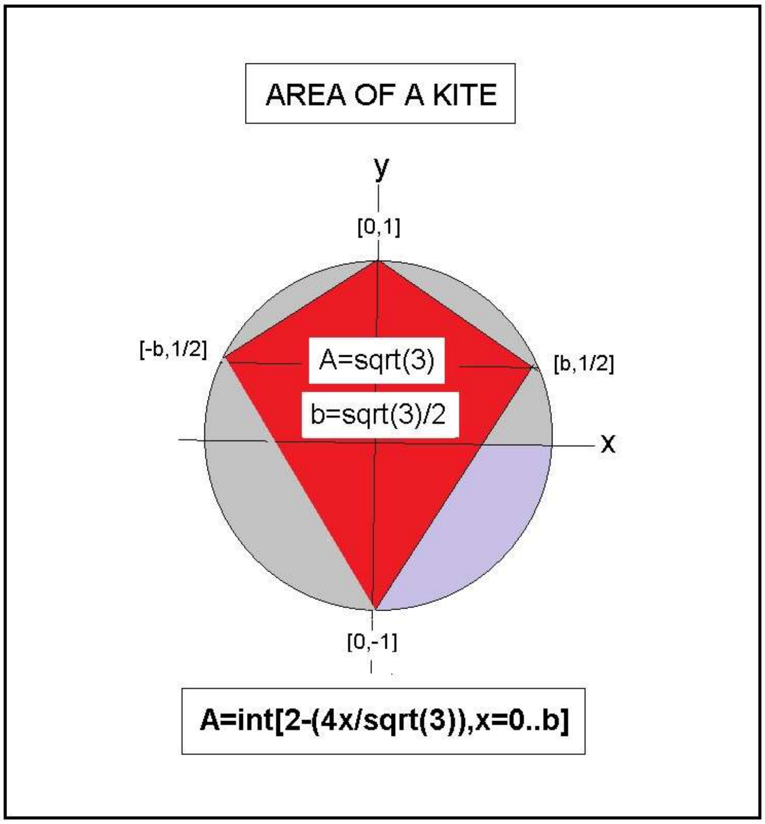
We want to find the area of the small blue region. The simplest way to get this is to take the area of the central equilateral triangle shown in red and subtract one sixth the area of each of the three circular areas. Using simple mathematics, this yields-

$$A= \text{sqrt}(3)-3\pi/6=\text{sqrt}(3)-\pi/2=0.161254\ldots$$

To confirm this answer, we can also use the calculus approach. It produces the following rather complicated integral with the same result-

$$A=2\int_{x=-1/2}^{o}[sqrt(3)-sqrt(1-x^2)-sqrt(-2x-x^2))]dx = \text{sqrt}(3)-\pi/2$$

As the next problem involving an area formed by a circle and inscribed straight lines , consider the following circle-kite configuration-



Here we want to find the area of a kite shown in red with the four vertex points indicated. The circle in grey has a radius of r=1. Using the Pythagorean Formula, we find that the short and long side-lengths of the kite are-

l=sqrt(3/4+1/4)=1 and L=sqrt(9/4+3/4)=sqrt(3).

So the area of the kite will be twice the area of the large right triangle lL/2 . Hence we have-

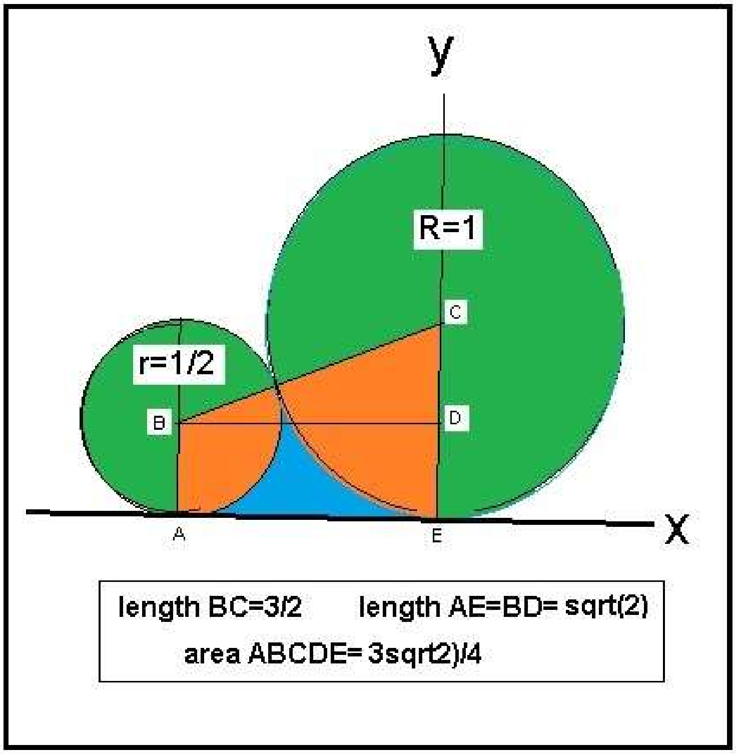
A=1*sqrt(3)=sqrt(3)

To confirm this answer we can again employ calculus to yield the integral-

$$A=2\int_{x=0}^{b}[2-\frac{4x}{sqrt(3)}]\,dx = sqrt(3)$$

On integrating, using b=sqrt(3)/2, produces the same result for A.

As a final example of finding an interior area consider the blue area formed by two touching circles of radii 1 and ½ and the x axis. Here is the picture –



Here the hypotenuse of the central triangle has length L=3/2 and base BD=sqrt(2) according to Pythagoras. The total area ABCDE=3sqrt(2)/4. Also the smallest angle of the triangle equals θ=arcsin(1/3)=19.471deg=0.33983rad. The blue sub-area equals 3sqrt(2)/4 minus the orange area. That is-,

$$A_b=3\text{sqrt}(2)/4-5\pi/16+(3/8)\arcsin(1/3)=0.206351\ldots$$

To confirm this answer we use alternate integral approach. We have after some manipulations that-

$$A_b=\text{int}\{-\text{sqrt}[y(2-y)]+\text{sqrt}(2)-\text{sqrt}[y(1-y)],\ y=0..2/3\}$$

On integrating we get the same area for the blue region but with a lot more work.

What we have shown with the above three examples is that it possible to calculate the areas of any sub areas constructed by circles and straight lines. Only a knowledge of elementary math through trigonometry is needed. The validity of the solutions can always be tested via standard integral calculus. The calculus approach tends to be a bit more involved.

U.H.Kurzweg
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Gainesvile, Florida