DERIVATION OF THE SECOND LINEARLY INDEPENDENT SOLUTION OF THE BESSEL EQUATION FOR INTEGER ORDER

We have shown in class that the complete solution of the Bessel equation for non-integer order ν is-

$$y(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x)$$

However, when ν becomes an integer n the second solution is no longer linearly independent of the first since $J_{-n}(x)=(-1)^n J_n(x)$. Thus one needs to do something else. The first thought is to use the Abel identity which states that a second linearly independent solution should be-

$$y_2(x) = J_\nu(x)\int_0^\infty \frac{dx}{xJ_\nu(x)^2}$$

The difficulty with this result is that the infinite series for the Bessel function of the first kind enters as a square in the denominator of the Abel integral and hence makes evaluation extremely cumbersome. To avoid such a complication one rather defines(as first done by Weber) the second linearly independent solution as the indeterminate ratio-

$$Y_\nu(x) = \frac{[J_\nu(x)\cos(\pi x) - J_{-\nu}(x)]}{\sin(\pi x)}$$

evaluated as ν->n. Applying the L'Hospital rule, one obtains the equivalent form-

$$Y_\nu(x) = (1/\pi)[\partial J_\nu(x)/\partial\nu - (-1)^\nu \partial J_{-\nu}(x)/\partial\nu]$$

Now, using the infinite series definition for $J_\nu(x)$, one finds that-

$$\frac{\partial J_\nu(x)}{\partial\nu} = \ln(x/2)J_\nu(x) - \sum_{n=0}^\infty \frac{(-1)^k (x/2)^{2k+\nu}\,\Psi(k+\nu+1)}{k!\Gamma(k+\nu)!}$$

and-

$$\frac{\partial J_{-\nu}(x)}{\partial \nu} = -\ln(x/2)J_{-\nu}(x) + \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2n-\nu}\Psi(k-\nu+1)}{k!(k-\nu)!}$$

where **Ψ(z)=d(ln(Γ(z))/dz** is the digamma function(see **Abramowitz and Stegun**) with Γ**(z)** being the standard Gamma function.

Substituting these last two partial derivative terms into the above equation for $Y_v(x)$, we find that the second linearly independent solution for integer n is-

$$Y_n(x) = \left(\frac{2}{\pi}\right)J_n(x)\ln(x/2) - \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k (x/2)^{2k+n}\Psi(k+n+1)}{k!\,(k+n)!}$$
$$-\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k (x/2)^{2k-n}\Psi(k-n+1)}{k!\,(k-n)!}$$

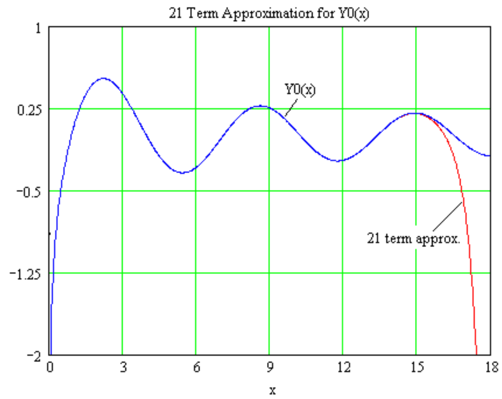
For the special case of n=0 this result reduces to-

$$Y_0(x) = \left(\frac{2}{\pi}\right)\left[J_0(x)\ln(x/2) - \sum_{k=0}^{\infty}\frac{(-1)^k\ (x/2)^{2k}\ \Psi(k+1)}{k!\,k!}\right]$$

Note that in the literature one usually finds that the expression for $Y_n(x)$ contains the Euler-Mascheroni constant γ This constant can be made to appear in the above expressions by making use of the identity-

$$\Psi(1+z) = -\gamma + \sum_{k=1}^{\infty}\frac{z}{k(k+z)}$$

but is not done here because it makes the expression for $Y_n(x)$ even more cumbersome than it already is. To convince you that the above expression for $Y_0(x)$ is correct, I plot here the approximation obtained by summing from k=0 to k=20.



Note that there is excellent agreement with the built in function for $Y_0(x)$ up to about x=15. Taking more terms in the series will progressively extent this range to larger values of x.

An alternative way to generate the function $Y_\nu(x)$ is to go to the Weber expression, used as our starting point in the present development, and look at things very close to an integer such as $\nu=n+\epsilon$ where , say, $\epsilon=0.001$. Doing this gives excellent approximations for $Y_n(x)$. It does, however, require that you have to generate your own Bessel function of the First Kind of non-integer order unless these happen to be already built into your program library.