# The Brachistochrone Problem of Johann Bernoulli

In 1696 Johann Bernoulli of Basel proposed the mathematical challenge in a mathematical journal of finding the particular curve along which a bead sliding without friction from rest at point A to a second point B(not directly beneath A) will take the minimum amount of time. There were a total of five correct responses received by the time of an imposed deadline . The answers were by Johann himself, Leibnitz, Newton, de l'Hospital, and Johann's brother Jacob. As Johann first stated, the curve is the cycloid of Huygens. Let's look at the complete solution here using the calculus of variations..

The time of transit for the bead from point A to B is-

$$T = \int_A^B \frac{dx}{V} = \int_0^{\pi/2} \frac{\sqrt{1+y'(x)^2}}{\sqrt{-2gy(x)}}dx = \int_0^{\pi/2} F(y,y')dx$$

where one has used the conservation of energy law (½)V^2+gy(x)=0 by placing the starting point A at x=y=0 and the ending point B at x=π/2, y=-1. We know from the calculus of variations that T will be minimized when F satisfies the Euler Lagrange equation-

$$\frac{d}{dt}\left[\frac{\partial F}{\partial y'}\right] - \frac{\partial F}{\partial y} = 0$$

For the specific form of F appearing here, one can integrate once to come up with-

$$y'\frac{\partial F}{\partial y'} - F = C$$

On letting $2gC^2=1$, this so-called brachistochrone problem reduces to finding a solution to the non-linear first order ODE-

$$(y')^2 = -\frac{(1+y)}{y}$$

subject to the initial condition y(0)=0. An exact solution of this last equation is easy to obtain. First let y= $-\sin^2(\theta/2)$. This converts the equation to-

$$\frac{d\theta}{dx} = \frac{1}{\sin^2(\theta/2)}$$

Which can be integrated directly to yield the classical parametric cycloid form-

$$x=(1/2)[\theta-\sin(\theta)] \qquad \text{and} \qquad y=-(1/2)[1-\cos(\theta)]$$

Note that θ=0 at the starting point A and has value θ=π at end point B where x= π /2 and y= -1.

The transit time of the bead sliding along the cycloid can now be obtained by noting that-

$$F=\frac{\sqrt{2/g}}{1-\cos(\theta)}$$

so that the elapsed time becomes-

$$T=\frac{1}{\sqrt{2g}}\int_0^{\pi} d\theta=\frac{\pi}{\sqrt{2g}}$$

To further verify that this indeed is the minimum time, take a few neighboring curves going through the same end points. In particular consider the straight line y=-(2/ π)x and the parabola $y=-1+(1-(2/\pi)x)^2$. A simple integration shows that in these cases the sliding times are 3.72419 /√(2g) and 3.27633 /√2g, respectively. Both are seen to be larger than the transit time along the brachistochrone. Another longer transit time occurs when the bead is first allowed to drop straight down and then continues sliding toward point B with its maximum acquired speed. For this case one finds T=[2+ (π /2)]/ √(2g. Another interesting property of the cycloid is that the bead transit time remains the same no matter where along the curve the bead starts from rest. Huygens used this principle as early as 1673 to design a better pendulum clock whose swing period is independent of its amplitude. Unfortunately the clock mechanism involved placing a cycloid trough at the attachment point of the pendulum wire which produced undesirable frictional effects on the pendulum swings.