CHARACTERISTICS AND THE CONVERSION TO CANONICAL FORM

Consider the second order PDE

$$a(x,y)z_{xx}+2b(x,y)z_{xy}+c(x,y)z_{yy}+F(z_x,z_y,z,x,y)=0$$

and introduce the characteristic variables η(x,y) and ξ(x,y). In terms of these variables the first partial derivatives become

$$z_x=\eta_x z_\eta+\xi_x z_\xi \quad , \quad z_y=\eta_y z_\eta+\xi_y z_\xi$$

A further application of the chain rule then leads to the second derivative terms

$$z_{xx}=\eta_x^2 z_{\eta\eta}+2\eta_x\xi_x z_{\eta\xi}+\xi_x^2 z_{\xi\xi}+\eta_{xx}z_\eta+\xi_{xx}z_\xi$$

$$z_{xy}=\eta_x\eta_y z_{\eta\eta}+(\eta_x\xi_y+\eta_y\xi_x)z_{\eta\xi}+\xi_x\xi_y z_{\xi\xi}+\eta_{xy}z_\eta+\xi_{xy}z_\xi\eta$$

$$z_{yy}=\eta_y^2 z_{\eta\eta}+2\eta_y\xi_y z_{\eta\xi}+\xi_y^2 z_{\xi\xi}+\eta_{yy}z_\eta+\xi_{xx}z_\xi$$

Substituting these into the above PDE yields a new equation with only a single second derivative term left after setting the coefficient multiplying the non-mixed second partial derivatives to zero. The resultant, so called, **canonical form** of our second order PDE is

$$B(\eta,\xi)z_{\eta\xi}=G(\eta,\xi,z_\eta z_\xi)+F(\eta,\xi,z_\eta,z_\xi,z)$$

where

$$B(\eta,\xi)=2a\eta_x\xi_x+2b(\eta_x\xi_y+\eta_y\xi_x)+2c\eta_y\xi_y$$

and

$$G(\eta,\xi,z_\eta,z_\xi) = a(\eta_{xx}z_\eta + \xi_{xx}z_\xi) + 2b(\eta_{xy}z_\eta + \xi_{xy}z_\xi) + c(\eta_{yy}z_\eta + \xi_{yy}z_\xi)$$

Here a, b c, and F are the terms appearing in the original PDE. Note that the condition for making the other two second partial derivative terms vanish is that the characteristic curves $\eta(x,y)$=constant and $\xi(x,y)$ = constant have the x and y dependent slope

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a}$$

Since the functions a, b and c are assumed to be real, we can catagorize the original equation by the sign of the radical. Thus the equation is termed **Hyperbolic** when **b^2-ac>0**. It is **Elliptic** when **b^2-ac<0** and **Parabolic** when **b^2-ac=0**. Note that hyperbolic equations have two families of real chracteristics while elliptic equations have no real characteristics.