ON THE SERIES EXPANSIONS OF K(k) AND E(k)

To obtain the infinite series representations for the complete elliptic integrals of the first and second kind we begin with the basic definitions-

$$K(k) = \int_{\theta=0}^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin(\theta)^2}} \quad and \quad E(k) = \int_{\theta=0}^{\pi/2} \sqrt{1-k^2\sin(\theta)^2}\, d\theta$$

On introducing the new variable transformation sin(θ)=tanh(z), one finds that-

$$K(k) = \int_{z=0}^{\infty} \frac{dz}{\cosh(z)\sqrt{1-\Delta^2}} \quad and \quad E(k) = \int_{z=0}^{\infty} \frac{\sqrt{1-\Delta^2}}{\cosh(z)} dz$$

where Δ=k tanh(z). On expanding the radicals one finds the infinite series expansions-

$$K(k) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{[1.3.5..(2n-1)]k^{2n}}{2^n n!} \int_{z=0}^{\infty} \frac{\sinh(z)^{2n}}{\cosh(z)^{2n+1}} dz$$

$$= \frac{\pi}{2}[1 + \sum_{n=1}^{\infty} \frac{[(2n)!]^2 k^{2n}}{2^{4n}(n!)^4}]$$

and-

$$E(k) = \frac{\pi}{2}[1 - \frac{k^2}{4}] - \sum_{n=2}^{\infty} \frac{[1.3.5..(2n-3)]k^{2n}}{2^n n!} \int_{z=0}^{\infty} \frac{\sinh(z)^{2n}}{\cosh(z)^{2n+1}} dz$$

$$= \frac{\pi}{2}[1 - \sum_{n=1}^{\infty} \frac{[(2n)!]^2 k^{2n}}{(2n-1)2^{4n}(n!)^4}]$$

These series are rapidly convergent for small k. For the intermediate value of k=1/sqrt(2) one finds-

$$K(\frac{1}{\sqrt{2}})=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{[(2n)!]^2}{2^{5n}(n!)^4}=\frac{\pi}{2}[1+\frac{(2!)^2}{2^5(1!)^4}+\frac{(4!)^2}{2^{10}(2!)^4}+...]$$

and-

$$E(\frac{1}{\sqrt{2}})=\frac{\pi}{2}[1-\sum_{n=1}^{\infty}\frac{[(2n)!]^2}{(2n-1)2^{5n}(n!)^4}]$$

$$=\frac{\pi}{2}[1-\frac{(2!)^2}{1(2^5)(1!)^4}-\frac{(4!)^2}{3(2^{10})(2!)^4}-\frac{(6!)^2}{5(2^{15})(3!)^4}-...]$$

These last two series can be used to calculate π to any desired degree of accuracy by using the Legendre identity(see Abramowitz and Stegun) which , for k=1/sqrt(2), reads-

$$\pi=2K(\frac{1}{\sqrt{2}})[2E(\frac{1}{\sqrt{2}})-K(\frac{1}{\sqrt{2}})]$$

A little manipulation shows this last result to be equivalent to-

$$\frac{2}{\pi}=\{1+\sum_{n=1}^{N}S(n)\}\{1-\sum_{n=1}^{N}S(n)[\frac{(2n+1)}{(2n-1)}]\}$$

where $S(n)=[(2n)!]^2/2^{5n}(n!)^4$and N->∞. For a ten place accuracy of π in this last expression one needs to take at least the first thirty terms (N=30) . To get around this relatively slow convergence one can directly evaluate the integrals for K(1/sqrt(2)) and E(1/sqrt(2)) by the AGM method of Gauss. If one does this and then substitutes into the above Legendre identity, π can readily be found to a billion place accuracy.