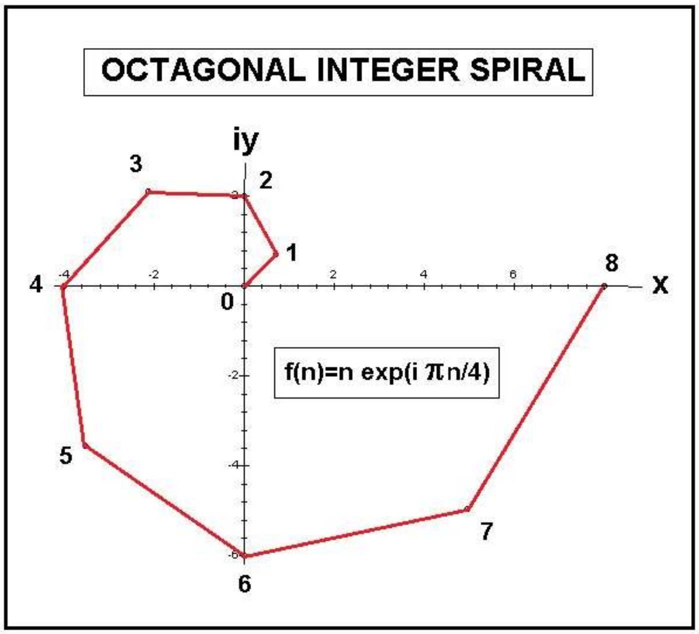
PROPERTIES OF INTEGER SPIRALS

About twenty five years ago while teaching a course in applied mathematics to our undergraduate engineering majors here at the University of Florida (https://mae.ufl.edu/~uhk/ANALYSIS.html) we came across the complex point function-

$$f(n)=n \exp(i\pi n/4)$$

, where n=0,1,2,3,4,… On plotting these points in the complex plane and connecting them with straight lines on arrives at a new spiral like structure shown-



I call this figure an octagonal integer spiral. It has exactly eight vertexes per one turn of the spiral. The external angle shift between two neighboring straight lines equals π/4 radians. Note the labelling of the vertexes as ascending positive integers n=0,1,2,3,.. The distance from the origin to any vertex n just equals n.

We wish in this note to generalize the above octagonal integer spiral form into one having 2m-tuple vertexes per turn. In particular we will be interested in the six vertexes per turn spiral, since it will lead to an interesting and greatly simplified version of the Ulam Spiral.

We start with a generalized integer spiral with 2m vertexes per spiral turn. This has the form-

$$f(n,m)=n \exp(i\pi n/m)=n\{\cos(i\pi n/m)+i\sin(i\pi n/m)\}$$

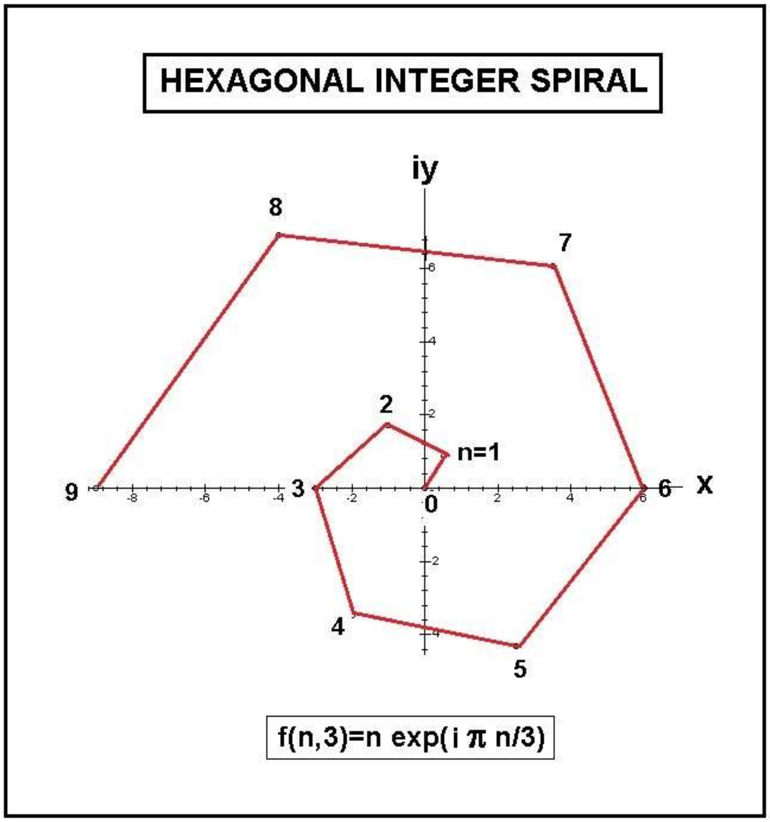
with π/m being the first angle between the x axis and the unit length line for n=1.This angle should be an integer fraction of 2π so that the spiral vertexes coincide. Let us plot the integer spiral-

$$f(n,3)=n\ \exp(i\pi n/3)=n\{\cos(\pi n/3)+i\sin(\pi n/3)\}$$

It will produce an integer spiral with six vertexes with the polar angle between n=0 and n=1 in the first quadrant of the complex plane being π/3 radians. Here are the point values for f(n,3) up to n=9-

n	f(n,3)
0	0
1	[1+i sqrt(3)]/2
2	-1+i sqrt(3)
3	-3
4	-2[1+i sqrt(3)]
5	(5/2)[1-i sqrt(3)]
6	6
7	(7/2)[1+i sqrt(3)]
8	4[-1+i sqrt(3)]
9	-9

Connecting neighboring points with straight lines produces the following six vertexes per turn spiral shown-



Note that the distance between n and n+6 is always six units. This means that the prime number 7 corresponds to 13 on the next spiral turn followed by 19 on the next turn. Likewise the prime 5 is followed by 11 and then followed by 17 on subsequent spiral turns. Writing down all these primes we get the prime set-

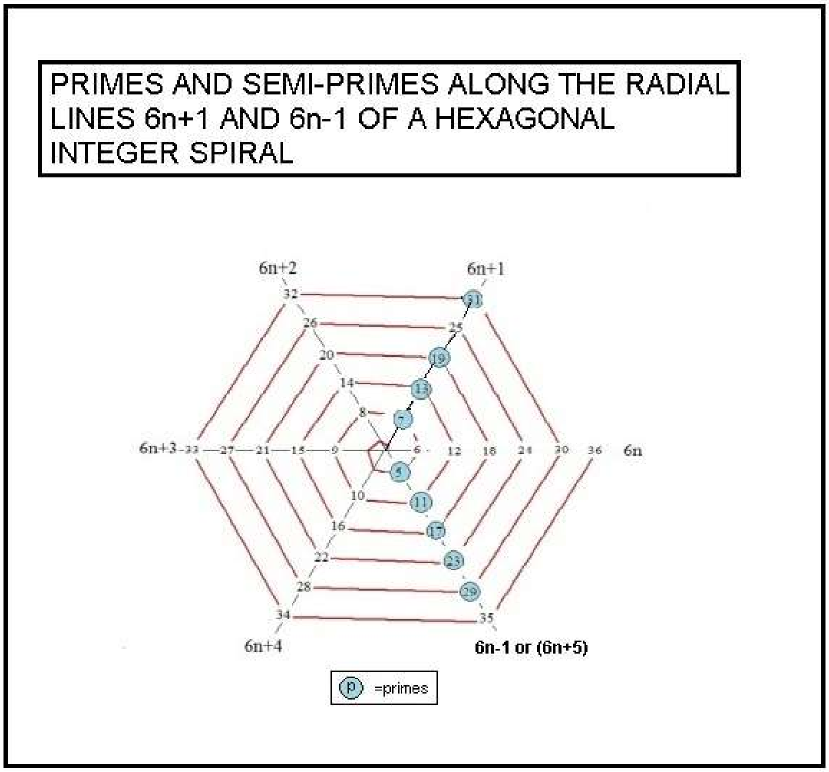
$$S=\{5,7,11,13,17,19,..\}$$

These happen to be all primes starting with five and are located at the intersection of just two radial lines $6n\pm1$ and the spiral vertexes corresponding to Nmod(6)=1 or 5.

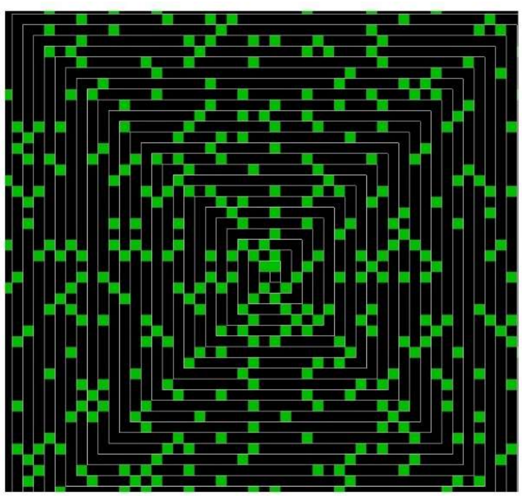
An expanded hexagonal integer plot first found by us about twenty years ago

https://mae.ufl.edu /~uhk / MORPHING-ULAM.pdf

follows-



The blue circles locate the prime numbers with the gaps at 25 and 35 representing composite numbers. This way of presenting prime numbers is far superior to the way primes have been plotted in the past. The standard Ulam Spiral yields the much more complicated form for primes as shown-



Here the green squares show the prime location as found, for example, at
https://www.alpertron.com.ar/ULAM.HTM

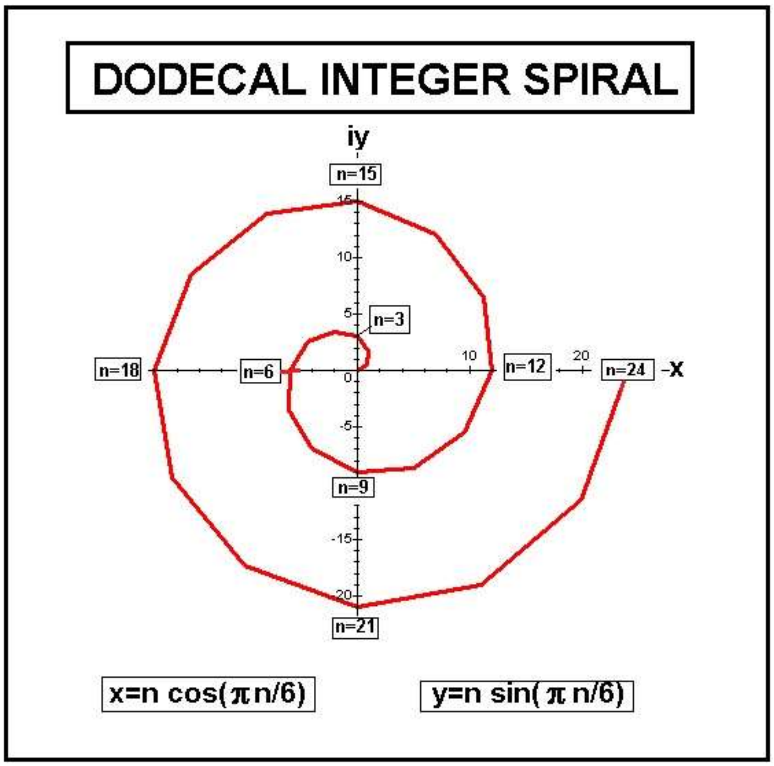
As the last of the N-Tuple Integral Spirals, let us consider the twelve vertex Spiral defined as-

$$f(n,6)=n\exp(i\pi n/6)= n\cos(\pi n/6)+i\, n\sin(\pi n/6)$$

We can easily plot this point function with connecting straight lines to produce a multiple turn integer spiral using the following one line MAPLE program-

listplot([seq([n*cos(n*Pi/6),n*sin(n*Pi/6)], n=0..24)],scaling=constrained, thickness=3,color=red);

Here is the plot-

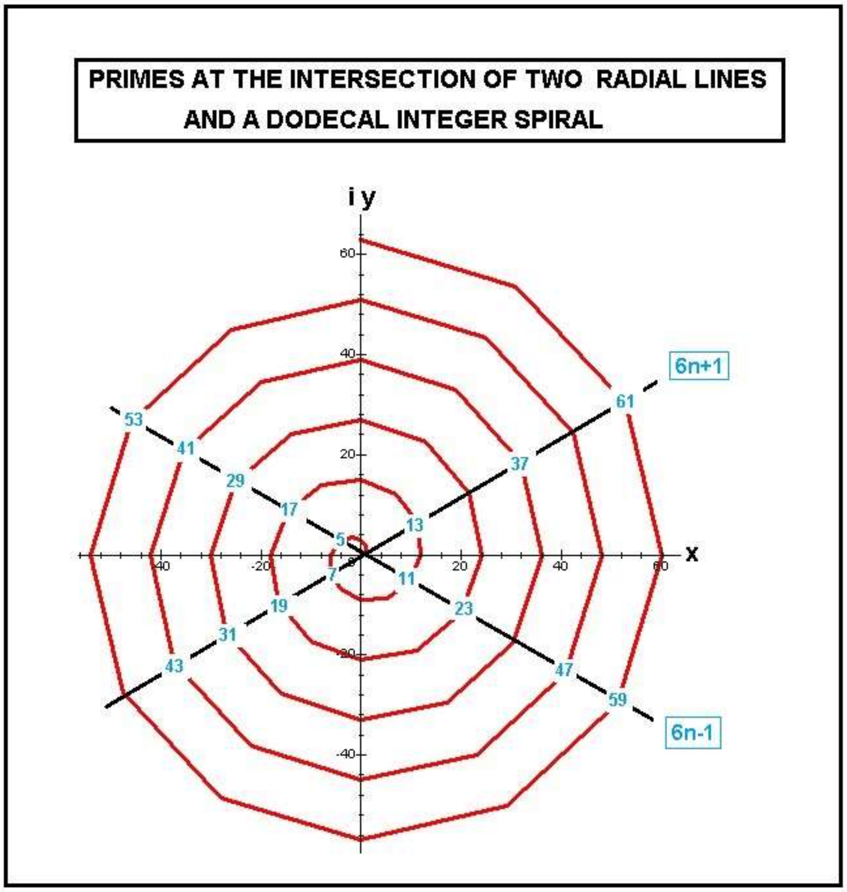


This Dodecal Integer Spiral is related to the earlier Hexagonal Spiral in that it can be used to locate primes. The spacing between subsequent turn of its spiral differs by twelve units. There are just two radial lines emanating from the origin and passing through a vertex along which primes may be present. These lines are located at thirty degrees away from the x axis. They are given by –

$$y=\pm x/\text{sqrt}(3)$$

The lines intersects the vertex points at 12n+1 and 12n-1.

Here is a picture which shows these two radial lines and how they intersect the Dodecal Integer Spiral-



Note that there are cases along these two radial lines where the intersection with a spiral vertex will not always yield a prime. Cases such as N=25 and 35 come to mind. An interesting observation concerning this Integer Spiral representation is that twin primes can be read off directly. We see that [41,43], [29,31], [17,19], [5,7], [13,11], and [61,59] are twin primes.

A final observation is that as m gets large the distance between neighboring vertex points goes toward zero. In this limit the figure approaches an Archimedes Spiral $r= a\theta$.

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