USE OF COMPLEX VARIABLE METHODS TO FIND TRIGNOMETRIC IDENTITIES

Many trigonometric Identities may be derived without much effort by use of complex variable methods. A starting point is the well known Euler Identity-

$$\exp(iz) = \cos(z) + i\sin(z)$$

If we let z=z_1+z_2+z_3+ …+ z_N one has that-

$$\cos(\sum_{n=1}^{N} z_n) = \mathrm{Re}\left[\prod_{n=1}^{N}[\cos(z_n + i\sin(z_n))]\right]$$

and

$$\sin(\sum_{n=1}^{N} z_n) = \mathrm{Im}\left[\prod_{n=1}^{N}[\cos(z_n) + i\sin(z_n)]\right]$$

Thus when z_1=A and z_2=B with N=2, we have-

$$\cos(A+B) = \mathrm{Re}[(\cos(A) + i\sin(A))(\cos(B) + i\sin(B))]$$
$$= \cos(A)\cos(B) - \sin(A)\sin(B)$$

and

$$\sin(A+B) = \mathrm{Im}[(\cos(A) + i\sin(A))(\cos(B) + i\sin(B))$$
$$= \sin(A)\cos(B) + \sin(B)\cos(A)$$

The double angle formulas –

$$\cos(2A) = 2\cos(A)^2 - 1 \quad and \quad \sin(2A) = 2\sin(A)\cos(A)$$

follow after setting A=B. Also setting 2A=B in these last results, we have the half angle formulas-

$$\cos(\frac{B}{2}) = \sqrt{\frac{1+\cos(B)}{2}} \quad and \quad \sin(\frac{B}{2}) = \sqrt{\frac{1-\cos(B)}{2}}$$

Also-

$$\tan(A+B)=\frac{\mathrm{Im}[(\cos(A)+i\sin(A))(\cos(B)+i\sin(B))]}{\mathrm{Re}[(\cos(A)+i\sin(A))(\cos(B)+i\sin(B))]}=\frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$$

Working out cos(6A) with aid of my computer, we find-

$$\cos(6A)=\mathrm{Re}(\prod_{n=1}^{6}(\cos(A)+i\sin(A))=32\cos(A)^6-48\cos(A)^4+18\cos(A)^2-1$$

Trignometric identities involving hyperbolic functions follow from the definitions-

$$\sinh(z)=\frac{e^z-e^{-z}}{2}=-i\sin(iz)\quad and\quad \cosh(z)=\frac{e^z+e^{-z}}{2}=\cos(iz)$$

Plugging into the above double angle formulas we find-

$$\cosh(A+B)=\cosh(A)\cosh(B)+\sinh(A)\sinh(B)\quad and$$
$$\sinh(A+B)=\sinh\left(A\right)\cosh(B)+\sinh(B)\cosh(A)$$

Next we use the log identity-

$$\ln\prod_{n=1}^{N}(A_n+iB_n)^{p_n}=\sum_{n=1}^{N}p_n\ln(A_n+iB_n)$$ **n**

where A_n, B_n, p_n are real. Using $\ln(A_n+iB_n)=\ln(\mathrm{sqrt}(A_n^2+B_n^2))+i\arctan(B_n/A_n)$, the imaginary part of this equality reduces to-

$$\arctan(\frac{\mathrm{Im}(G)}{\mathrm{Re}(G)})=\sum_{n=1}^{N}p_n\arctan(\frac{B_n}{A_n})\quad with\quad G=\prod_{n=1}^{N}(A_n+iB_n)$$

There are an infinite number of arctan relations which arise from this last result. One of the simplest is-

$$\arctan(\frac{a+b}{ab-1})=\arctan(\frac{1}{a})+\arctan(\frac{1}{b})$$

obtained by letting A_1=-a, B_1=1, A_2=b, B_2=1, p_1=p_2=1. One can recover the famous two term arctan formula for π due to Machin by letting A_1=5, B_1=1, A_2=239, B_2=1 and p_1=4 , p_2=-1. It reads-

$$\frac{\pi}{4} = \arctan(1) = 4\arctan(\frac{1}{5}) - \arctan(\frac{1}{239})$$

Finally we have that –

$$\arctan(z) = \int_{u=0}^{z} \frac{du}{1+u^2} = \frac{1}{2i}\int_{u=0}^{z} \frac{du}{i+u} - \frac{1}{2i}\int_{u=0}^{z} \frac{du}{-i+u} = \frac{1}{2i}\ln(\frac{z-i}{z+i}) + \frac{\pi}{2}$$

so that arctan(1/5)=(1/(2i)*ln((1-5i)/(1+5i))+π/2=0.197395…