CONTINUED FRACTIONS

A general continued fraction is defined as-

$$F = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \dots}}}$$

with its convergents (gotten by taking up to the nth term in the expansion)given by-

$$F_1 = a_0 \quad , \quad F_2 = \frac{a_0 a_1 + b_1}{a_1} \quad , \quad F_3 = \frac{a_0 a_1 a_2 + a_0 b_2 + b_1 a_2}{a_1 a_2 + b_2} \quad , \quad etc$$

For the special cases of $b_1=b_2=b_3= b_n=1$ with $a_0=1$ and $a_1=a_2=a_3=a_4=a_n=2$ one finds the convergents to be $F_1=1$, $F_2=3/2$, $F_3=7/5$, $F_4=17/12=1.416666$. These numbers clearly converge to the value sqrt(2)=1.41421356… so that one can write-

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}$$

There are an infinite number of other continued fractions possible with many of the more interesting ones having been found quite early by mathematicians such as Euler, Lagrange, and Lambert. It should be noted that a given number or function F can have more than one continued fraction expansion as we will demonstrated below. Also we point out that many computer programs such as MAPLE, MATHEMATICA, and MATLAB have the built-in capability to generate simple continued fractions .

As a starting point for generating a continued fraction, we consider representing the square root of the number N. To get the continued fraction in this case we begin with-

$$(\sqrt{N} - \sqrt{N_0})(\sqrt{N} + \sqrt{N_0}) = N - N_0$$

or its equivalent form-

$$\sqrt{N} = \sqrt{N_0} + \frac{N - N_0}{\sqrt{N_0} + \sqrt{N}}$$

Next re-substitute this value for sqrt(N) repeatedly into the expression to obtain the infinite general continued fraction-

$$\sqrt{N} = \sqrt{N_0} + \cfrac{(N-N_0)}{2\sqrt{N_0} + \cfrac{(N-N_0)}{2\sqrt{N_0} + \cfrac{(N-N_0)}{2\sqrt{N_0} + \dots\dots}}}$$

which will converge to the correct square root value provided $N-N_0$ is small enough. For the case of N=2 and $N_0=1$ one recovers the expansion given above. If, however one is looking for a more rapidly converging continued fraction, the result-

$$\sqrt{2} = 1.4 + \cfrac{(0.04)}{(2.8) + \cfrac{(0.04)}{(2.8) + \cfrac{(0.04)}{(2.8) + \dots}}}$$

is much better. Here $2a_0=a_1=a_n=14/5$ and $b_n=1/25$. One can also produce continued fraction expansions for functions G(x) based on their infinite series expansions. Take the case of-

$$G(x) = (\frac{1}{x})\arctan(x) = 1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \dots.$$

We start the continued expansion by noting a convergents expansion can only contain terms of the form x^{2n} and the a_ns should contain the odd integers 2n+1. We thus try the form-

$$G(x) = \cfrac{1}{1 + \cfrac{A_1 x^2}{3 + \cfrac{A_2 x^2}{5 + \cfrac{A_3 x^2}{7 + ..}}}}$$

On inverting the arctan series we find that a comparison with the second convergents yields-

$$1 + \frac{x^2}{3} - \frac{4x^2}{45} + O(x^6) = 1 + \frac{A_1 A_3 x^2}{3A_3 + A_2 x^2} \cong 1 + \frac{A_1 x^2}{3}[1 - \frac{A_2}{A_3}x^2 + O(x^4)]$$

From this we see that A_1=1 and A_2=4. Continuing the procedure, one finds A_3=9, A_4=16, and eventually $A_n=n^2$. So one obtains the convergent partial fraction -

$$\left[\frac{1}{x}\right]\arctan[\ x] = \cfrac{1}{1+\cfrac{(1x)^2}{3+\cfrac{(2x)^2}{5+\cfrac{(3x)^2}{7+\ldots}}}}$$

already known to Leonard Euler. The smaller x becomes the more rapidly this fraction converges and thus is useful in evaluating certain arctan formulas for π where one deals with N>>1 in arctan(1/N) expansions. Note that for x=1/sqrt(3), one finds the continued fraction expression-

$$\frac{\pi\sqrt{3}}{6} = \cfrac{1}{1+\cfrac{(1/\sqrt{3})^2}{3+\cfrac{(2/\sqrt{3})^2}{5+\cfrac{(3/\sqrt{3}))^2}{7+....}}}}$$

which has a pleasant appearance but is not much good in actually determining the value of π to a large number of decimal places.

Other interesting continued fractions include-

$$\frac{e^2-1}{e^2+1} = \tanh(1) = 0+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{5+\ldots}}} = [0;1,3,5\ldots]$$

where the square bracket notation is often used for simple continued fraction abbreviations. Also-

$$\ln(2) = \cfrac{1}{1+\cfrac{1^2}{1+\cfrac{2^2}{1+\cfrac{3^2}{1+\ldots}}}}$$

which converges very slowly and, perhaps the simplest infinite continued fraction,-

$$\Phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1+}}} = [(1+\sqrt{5})/2] = 1.618033988...$$

which represents the ratio N_{n+1}/N_n of the Fibonacci sequence 1, 2, 3, 5, 8,.. as n goes to infinity and also equals the golden ratio. The proof follows from the Fibonacci number definition-

$$\Phi_{n+1} = \frac{N_{n+1}}{N_n} = \frac{N_n + N_{n-1}}{N_n} = 1 + \frac{1}{\Phi_n}$$

so that-

$$\Phi_{n+1} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\Phi_{n-2}}}}$$

which leads to the above result.

Another continued fraction having a very simple form is-

$$[\sqrt{n^2+1} - n] = \cfrac{1}{2n + \cfrac{1}{2n + \cfrac{1}{2n+}}}$$

This expansion arises in connection with solving the Diophantine equation $y^2 - K^2x^2 = 1$ and also in the expansion of sqrt(2) as shown above.

We finish our discussion on continued fractions by looking at some finite continued fractions which correspond to rational numbers. Look for example at the three digit number K=124=100+20+4. Here we can write K/100=1+(1/5)+(1/25). This tells us that one has-

$$\frac{K}{100} = 1 + \cfrac{1}{5 + \cfrac{1}{a_2 + \cfrac{1}{a_3 +}}}$$

with a_2 and a_3 to still be determined. Writing down the second convergent, we have

$$\frac{(K-100)}{100} = \frac{a_2}{(5a_2+1)}$$

which solves as a_2=-(1+(1/5)) so that one chooses a_2= -1 as the nearest integer. Next taking the third convergent one finds a_3= -5 exactly so that the continued fraction terminates as

$$\frac{K}{100} = 1 + \cfrac{1}{5 + \cfrac{1}{-1 + \cfrac{1}{-5}}} = [1; 5, -1, -5]$$

One can also use canned mathematics programs to quickly obtain simple continued fractions. For example, the two line MAPLE command- *with(numtheory)* followed by *cfrac(Pi, 6)*- produces a simple continued fraction for Pi good through six convergents. The result reads [3;7,15,1,292,1,1,]. Note the non-regularity in the numbers for this irrational number. The large number 292 indicates that a very good approximation will be achievable by terminating things at the third convergent. This produces-

$$\pi \cong 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}} = \frac{355}{113} = 3.1415929..$$

which is accurate to six decimal places. The approximation is referred to as the Otto ratio(after Valentin Otto, Professor of Astronomy, Wittenberg, 1573) although it was already known to Chinese mathematicians a thousand years earlier.

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