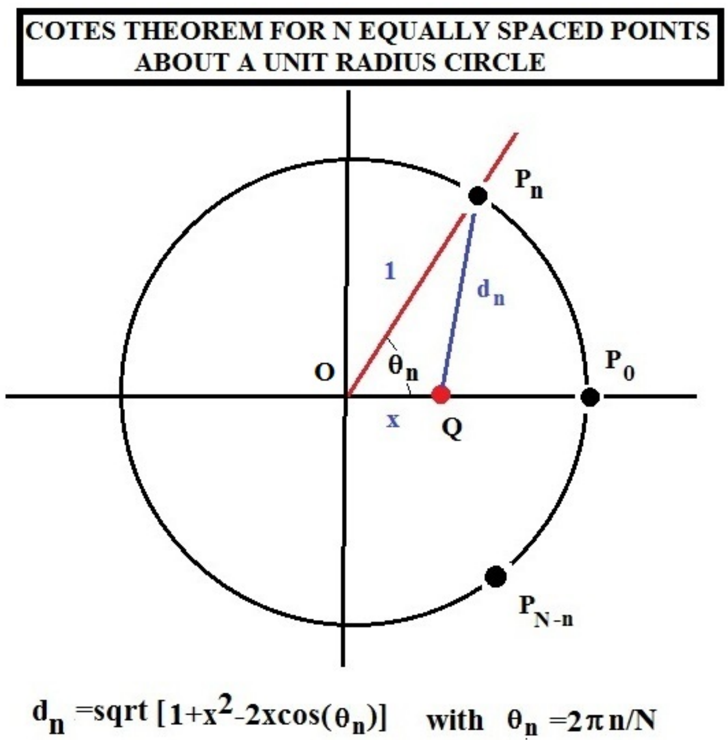
PROOF OF THE COTES THEOREM

Almost 300 years ago Roger Cotes(1682-1716), a contemporary of Newton at Cambridge, came up with a theorem in which he stated that-

The product of the distances from equally spaced points along a unit radius circle to a point Q located along the x axis at distance x from the circle origin O and on a line passing through the first point $P_0$ on the circle equals $1-x^N$

A picture of the problem follows-




If we have a total of N equally spaced points on the circle then the angle between neighboring points measured from the origin O is $2\pi/N$ and point $P_n$ has the polar coordinates $[r,\theta]=[1,2\pi n/N]$. Point $P_0$ is located on the x axis with polar coordinates [1,0]. Now some simple geometry shows that the distance $QP_n$ is-

$$d_n = \sqrt{(x-\cos(\theta_n))^2+\sin(\theta)^2} = \sqrt{(1+x^2-2x\cos(\theta_n)}$$

which is also expected from the law of cosines. Next we need to distinguish between even and odd N. Taking first the even N case, we see from the above picture that $d_n=d_{N-n}$ . So the product of all the $d_n$s will be-

$$F(x,N)=(1+x)(1-x)\prod_{n=1}^{(N/2)-1}\{1+x^2-2x\cos(\frac{2\pi n}{N})\}$$

For the case of odd N , we find instead that-

$$F(x,N)=(1-x)\prod_{n=1}^{(N-1)/2}\{1+x^2-2x\cos(\frac{2\pi n}{N})\}$$

since odd N has no point lying along the negative x axis. Evaluating the cases N=3, 4, 5 and 6, we find-

$F(x,3)=(1-x)(1+x+x^2)$

$F(x,4)=(1-x)(1+x+x^2+x^3)$

$F(x,5)=(1-x)(1+x+x^2+x^3+x^4)$

$F(x,6)=(1-x)(1+x+x^2+x^3+x^4+x^5)$

From these results one can generalize to state that-

$$F(x,N)=(1-x)\sum_{n=0}^{N-1}x^n$$

Now we know that the incomplete geometric series-

$$S(x,N-1)=\sum_{n=0}^{N-1}x^n=\frac{1-x^N}{1-x}$$

Thus we find –

$$F(x,N)=1-x^N$$

which proves the Cotes Theorem. That Roger Cotes was able to come up with this result indicates that he was familiar with both the summation of finite series and understood the concept of generalization. It is unfortunate that he died at the early age of 33 from what appears to have been typhoid fever. As Newton remarked, had this young Cambridge professor lived one could have expected many additional mathematical discoveries by him.

There are many identities derivable from the Cotes Theorem. Among the more important is the following relation between a finite product and a finite series-

$$\prod_{n=1}^{\frac{N}{2}-1}\left\{(1+x^2)-2x\cos(\frac{2\pi n}{N})\right\}=\frac{1}{x(1+x)}\sum_{n=1}^{N}x^n \quad for\ N\ even$$

When N=6 and x=1/2 this result reads-

$$\sum_{n=1}^{6}\frac{1}{2^n}=(\frac{5}{4}-\frac{1}{2})(\frac{5}{4}+\frac{1}{2})(\frac{3}{4})=\frac{63}{64}$$

From the incomplete geometric  series we also find-

$$\sum_{n=0}^{N-1}\sin^{2n}(z)=\frac{1-\sin^{2N}(z)}{\cos^2(z)}$$

and –

$$\sum_{n=0}^{N-1}e^{-nz}=\frac{1-e^{-Nz}}{1-e^{-z}} \quad with \quad z>0$$

In addition one has that –

$$\sqrt[N]{-1}=\cos(\frac{\pi}{N})+i\sin(\frac{\pi}{N})$$

This anticipates the concept of a complex variable and also deMoivre's Formula for the root of a complex number.   The $6^{th}$ root of ( -1 ) will be [sqrt(3)+i]/2.

Note that Cotes Theorem applies equally well to N sided regular polygons inscribed in a unit radius circle. If the regular polygons lie in a circle of radius r=a, the Cotes Theorem is modified to-

$$F(x,a,N)=a^N-x^N$$

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