COUETTE FLOW VIA A SOLUTION OF THE BIHARMONIC EQUATION

Let us consider a low Reynolds number incompressible viscous flow created in the annular space between two concentric and co-rotating cylinders of infinite length. This problem is governed by the standard biharmonic equation expressed in cylindrical coordinates . The velocity field here is strictly in the theta direction and can depend only on the radial coordinate. Accordingly the Laplacian takes the form∇^2=(1/r)d/dr(rd/dr) and the expanded biharmonic equation for the flow streamline y becomes-

$$r^3 y'''' + 2r^2 y''' - r^2 y'' + ry' = 0$$

Noting that in cylindrical coordinates the velocity in the theta direction V=-dy/dr , this 4th order ODE may be expressed as –

$$r^3 V''' + 2r^2 V'' - rV' + rV = 0$$

This last equation is of the standard Euler type and is thus known to have solutions of the form V=r ^{n} .Substituting this into the above 3rd order ODE yields the algebraic expression (n-1)(n-1)(n+1)=0, so that the general solution becomes-

$$V = wr = r(A + B\ln r) + C/r$$

Where A, B, and C are arbitrary constants, w is the local angular velocity, and the fluid extends over the range a<r<b. Adjusting things to match the assumed constant angular velocities w_a and w_b at the cylinder walls, one finds that B=0, A= $[b^2w_b-a^2w_a]/[b^2-a^2]$ and $C=(ab)^2(w_a-w_b)/(b^2-a^2)$. This profile represents the classical Couette flow and has a shear stress of $\tau=\mu(A-C/r^2)$. Note that for a small gap where (b-a)/(b+a)<<1, the shear is essentially equal to the constant value $\tau=\mu(bw_b-aw_a)/(b-a)$. It is this last form for the shear stress which is often used to experimentally determine the viscosity coefficient μ of a liquid. Note that this Couette flow solution is strictly valid only for low Reynolds number flows and will become unstable against Taylor vortexes and/or turbulence at higher differential rotation rates of the cylinders.