SOLVING THE CUBIC EQUATION

The cubic algebraic equation $ax^3+bx^2+cx+d=0$ was first solved by Tartaglia but made public by Cardano in his book Ars Magna(1545) after being sworn to secrecy concerning the solution method by the former. The solution procedure is to first introduce the transformation $x=z-[b/(3a)]$. This produces the reduced cubic $z^3+Bz+C=0$ with no square term. Here $B=(c/a)-[b^2/(3a^2)]$ and $C=2[b/(3a)]^3-(bc)/(3a^2)+d/a$. Next let $z=(u-t)$ so that $(u-t)^3=u^3-t^3+3ut(t-u)$ by a simple expansion. Plugging into the reduced cubic, this yields $u^3-t^3+3ut(t-u)=-B(u-t)-C$ from which follows that $C=t^3-u^3$ and $B=3ut$. We thus find , upon eliminating t, that $u^6+Cu^3-(B/3)^3=0$. This is a simple quadratic equation in u^3 and yields the solution $u^3=\{-(C/2) \pm sqrt[(C/2)^2+(B/3)^3]\}$. So that we arrive at an analytic solution $x=u-B/(3u)-b/(3a)$ of the cubic. Ferrari, a pupil of Cardano, extended this type of procedure to solve the 4^{th} order algebraic equation in closed form. All attempts to find solutions to the quintic and higher failed and it was finally shown by Abel in 1824 that no general analytic solutions are possible for quintic equations and higher.

To demonstrate the solution method consider $x^3+2x-12=0$. Here a=1, b=0, c=2 and d=-12, so that B=2 and C=-12. Therefore $u^3=6 \pm sqrt(980/27)=6 \pm 6.024649761..$. Thus the positive root yields $u=2.290994..$ Substituting into the above form for x we find $x=2.290994-2/[3(2.290994)]-0=1.999999..$. That is, one root equals $x=2$ which checks via substitution into the original equation. Note that a peculiar property of this solution method is that the root does not become obvious until the last step in the procedure. The complex roots also follow by this approach, although both Cardano and Ferrari did not recognize them as such since complex numbers had not yet been invented. Today of course, we solve any order algebraic equation beyond second numerically by canned programs. Thus, for example, MAPLE gives

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solve(x^3+2*x-12=0, x);
2, -1 + I sqrt(5), -1 - I sqrt(5)
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for the three roots in the above example.