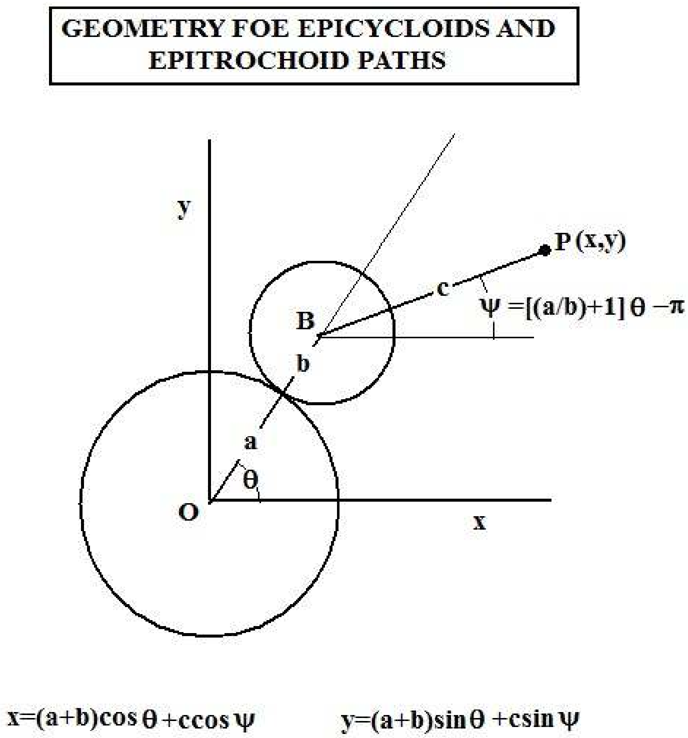
## EPICYCLOIDS, HYPOCYLOIDS, EPITROCHOIDS, AND HYPOTROCHOIDS

**There are four related classic curves which are generated by one circle rolling about the outside or inside of stationary second circle. The best known of these is the EPICYCLOID. It is generated by following the trajectory of a point P on the periphery of a rolling circle about a second stationary circle. When point P is located at distance c along a radial line coming from the center of the rolling circle but different from its radius, then the path generated is referred to as an EPITROCHOID. The derivation for these two curves is aided with help of the following diagram-**



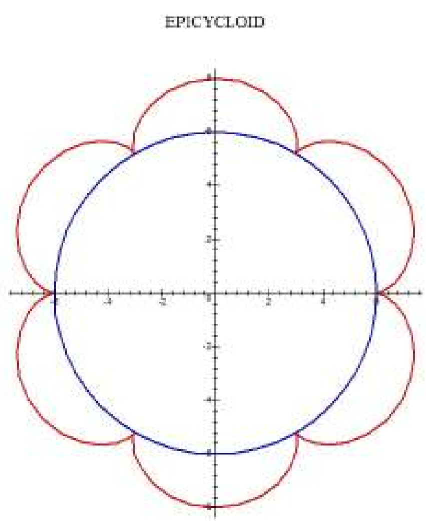

**Here we have a stationary circle of radius R=a and a second circle of radius R= b which rolls about the inner circle. A point P is located at distance c from the rolling circle center at B. The x and y coordinates of point P are -**

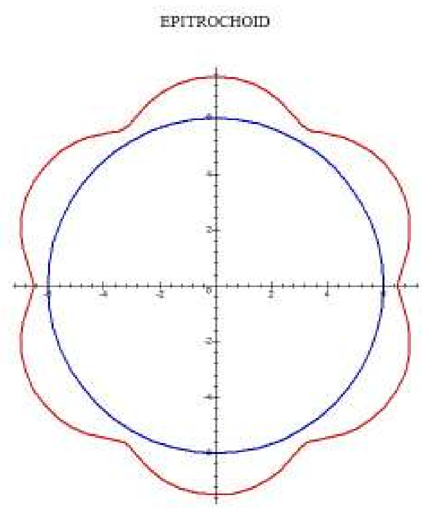
$$x_P = (a+b)\cos(\theta) + c\cos(\psi)$$
$$and$$
$$y_P = (a+b)\sin(\theta) + c\sin(\psi)$$

**A little geometry shows the angle ψ=(a/b+1)θ-π, so that cosψ=-cos(a/b+1)θ and sinψ=-sin(a/b+1)θ. Therefore the path traced out by point P will be-**

$$x_P = (a+b)\cos(\theta) - c\cos(a/b+1)\theta,$$
$$y_P = (a+b)\sin(\theta) - c\sin((a/b+1)\theta$$

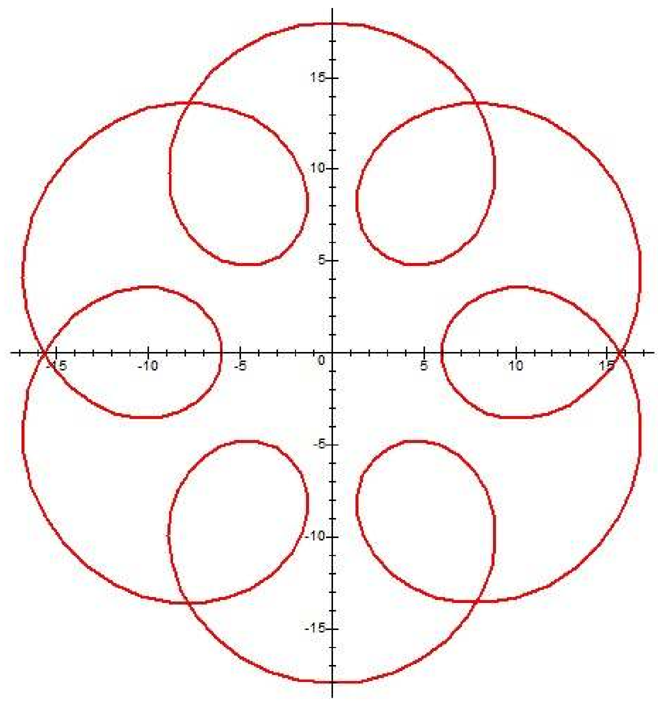
**This is the parametric representation of an EPICYCLOID when c=b and an EPITROCHOID when c does not equal b. We present here the epicycloid corresponding to a=6 , b=c=1 and the epitrochoid corresponding to a=6, b=1, and c=1/2-**




**The blue circles shown correspond to R=a=6 and are useful in distinguishing the cusp behavior of epicycloids compared to the smooth ripple of epitrochoids. By just varying a, b, and c an infinite number of other curves of this type can be generated. It is of historical interest that epicycloids and epitrochoids arise in connection with the Ptolemaic description of planetary motion and also in the design of Wankel engines. Here is another EPITROCHOID. This time we have a figure with six loops corresponding to a=12, b=2, and c=6-**

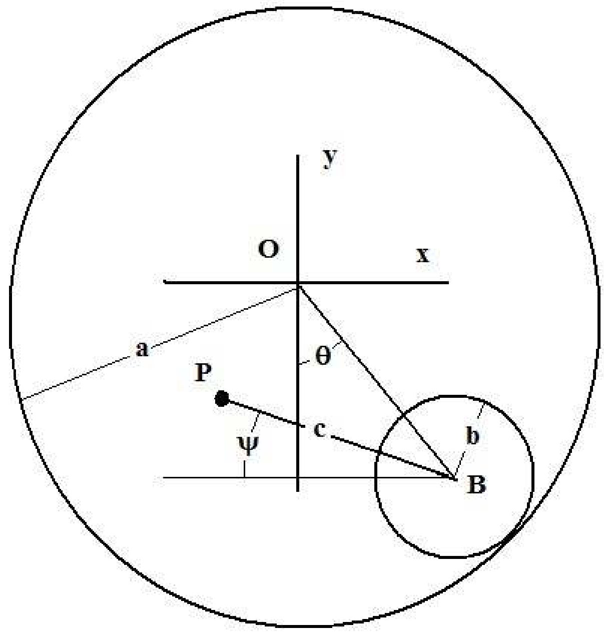
EPITROCHOID WITH SIX LOOPS




**a=12, b=2, and c=6**

**We look next at the figures generated by rolling the circle of radius R=b around the inside of the stationary radius R=a circle. Using the diagram shown it is straight forward to derive the trajectory of point P sitting along a radial line at distance c from the rolling circle center. We have-**

**GEOMETRY FOR HYPOCYCLOID AND HYPOTROCHOID PATHS**




$$x_P = (a-b)\sin(\theta) - c\cos(\psi) \ and \ y_P = -(a-b)\cos(\theta) + c\sin(\psi)$$

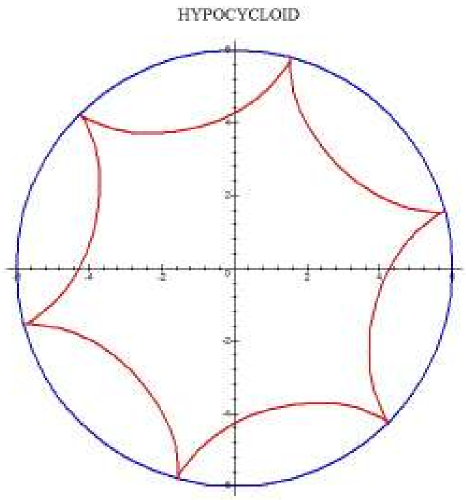
**Looking at the geometry of the figure one can deduce that $\psi=(a/b-1)\theta$, so that the parametric equations for the locus of P will be-**

$$x_P = (a-b)\sin(\theta) - c\cos(a/b-1)\theta$$

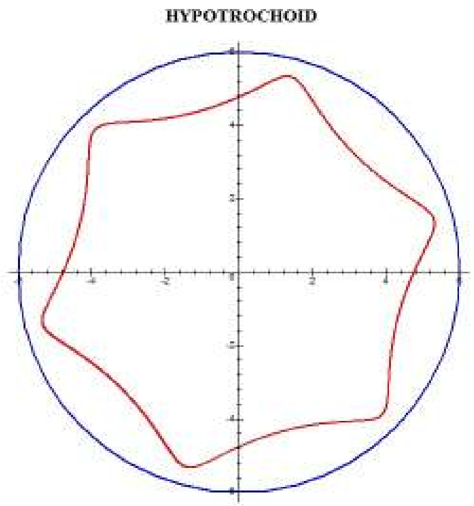
$$and$$

$$y_P = -(a-b)\cos(\theta) + c\sin(a/b-1)\theta$$

**These equations yield a HYPOCYCLOID when b=c and a HYPOTROCHOID when c differs from b. Here are some examples-**

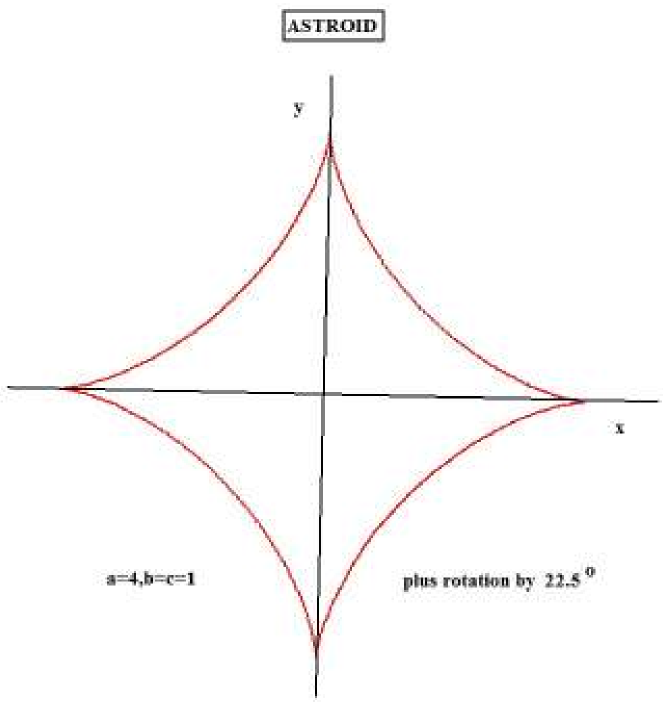



a=6, b=c=1




a=6, b=1, c=1/2

**An interesting special case of the hypercycloid is the ASTROID which occurs for a=4b=4c after a $\pi/8$ radian rotation. It has the four cusp form shown-**




**In this orientation the asteroid may be expressed in the much simpler parametric form-**

$$x = a\cos(t)^3 \quad and \quad y = a\sin(t)^3$$

**or as a single equation-**

$$x^{2/3} + y^{2/3} = a^{2/3}$$

**When a=1, the asteroid has the area A=3π/8 and a perimeter of P=6. The ASTROID was first examined by both Bernoulli and by Leibniz.**

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