PROPERTIES OF A MODIFIED ZETA FUNCTION

In a recent article on this web page we derived numerous identities involving the Zeta Function-

\[ \zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{with} \quad s = \sigma + i \tau \]

Two very important properties of this function are-

\[ \zeta(s) = \prod_{n=1}^{\infty} \frac{1}{1 - \frac{1}{p(n)^s}} = \frac{2^s}{(2^s - 1)} \sum_{n=1}^{\infty} \frac{1}{(2n - 1)^s} \]

,where \( p(n) \) represents the nth prime and the term in the denominator of the sum represents all odd integers. In looking at more details of this function it became clear to me there should also exist a related function defined as-

\[ \eta(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \quad \text{with} \quad s = \sigma + i \tau \]

Our purpose here is to determine some of the more important properties of this new zeta like function.

We begin by writing-

\[ \eta(s) = (1 + \frac{1}{3^s} + \frac{1}{5^s} + \ldots) - \frac{1}{2^s}(1 + \frac{1}{2^s} + \frac{1}{3^s} + \ldots) = \sum_{n=1}^{\infty} \frac{1}{(2n - 1)^s} - \frac{1}{2^s} \sum_{n=1}^{\infty} \frac{1}{n^s} \]

But we know the last two sums equal to the following –

\[ \sum_{n=1}^{\infty} \frac{1}{(2n - 1)^s} = \frac{2^s - 1}{2^s} \zeta(s) \quad \text{and} \quad \frac{1}{2^s} \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{2^s} \zeta(s) \]

Plugging in we get-

\[ \eta(s) = \left\{1 - \frac{1}{2^{s-1}}\right\} \zeta(s) \]

Plotting both \( \eta(s) \) and \( \zeta(s) \) over the range \(-9 < s < 10\) using our MAPLE program for which the Zeta function is known over much of the \( s = \sigma + i \tau \) plane, we find-
We see that both functions have zeros at the negative even integer values of $s$ and that-

$$\eta(1)=\pi/4=0.78539816\ldots \text{ while } \zeta(1)=\mp\infty,$$

$$\eta(\infty)=\zeta(\infty)=1$$

Also the Zeta function at $s=2$ was first shown by Leonard Euler to be $\pi^2/6$. Hence $\eta(2)=\pi^2/12$. We can generate a table of the Eta Function using the above relation between $\zeta(s)$ and $\eta(s)$. Here are the results for $-9<s<10$-

<table>
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<th>$\eta$</th>
<th>$s$</th>
<th>$\eta$</th>
</tr>
</thead>
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<td>1</td>
<td>0.78539=\pi/4</td>
</tr>
<tr>
<td>-8</td>
<td>0</td>
<td>2</td>
<td>0.82246=\pi^2/12</td>
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</table>

Using the earlier stated identity relating $p(n)$ to $\zeta(n)$, we also have the new identity-

$$\eta(s) = (1 - \frac{1}{2^{s-1}}) \prod_{n=1}^{\infty} \frac{1}{[1 - \frac{1}{p(n)^s}]}$$

For $s=2$, this yields-
\[
\eta(2) = \frac{1}{2} \left\{ \frac{4 \cdot 9}{3 \cdot 8} \cdot \frac{25}{24} \cdot \frac{49}{48} \cdot \frac{121}{120} \cdot \ldots \right\} = 0.82246\ldots
\]

We can also look at cases where \( s = \sigma + i\tau \) is complex. In that case the Eta Function is written as-

\[
\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^\sigma} \left\{ \cos[\tau \ln(n)] - i \sin[\tau \ln(n)] \right\}
\]

and will have a real and imaginary part. Take the case of \( s = \sigma + i\tau = 1 + i \). In this case-

\[
\eta(1 + i) = (1 - \frac{1}{2^2})\zeta(1 + i) = 0.7265597750 + i0.1580958640
\]

We could have gotten the same result using the above infinite series definition. Namely,-

\[
\eta(1 + i) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n)} \left( \cos(\ln(n)) - i \sin(\ln(n)) \right)
\]

Recalling from the Riemann study of zeros along the \( \sigma = 1/2 \) line in the \( s \) plane for the Zeta function, suggests to us that the Eta Function also may have some zeros not far removed from this line. To test this out we use a contour plotting approach to see if small closed circle contours appear. Such circles would indicate zeros of Eta(s).

We start with-

\[
A = \text{Re} \left\{ \left(1 - \frac{1}{2^{\sigma - i\tau - 1}}\right)\zeta(\sigma + i\tau) \right\} \quad \text{and} \quad B = \text{Im} \left\{ \left(1 - \frac{1}{2^{\sigma + i\tau - 1}}\right)\zeta(\sigma + i\tau) \right\}
\]

and then look at contours \( F=\text{const.} \) for-

\[
F = \sqrt{A^2 + B^2} = \text{Abs} \left\{ \left(1 - \frac{1}{2^{\sigma + i\tau - 1}}\right)\zeta(\sigma + i\tau) \right\}
\]

Looking insidea various rectangles in the \( s=\sigma+i\tau \) plane we find zeros along (1)the line \( \tau=0 \) for even negative \( \sigma \), (2) along the line \( \sigma=1/2 \) for the same zeros present for the Zeta function( \( \tau=14,13,21,02,25,01, \ldots \)) (3) plus zeros lying along the \( \sigma=1 \) axis starting with \( \tau=9.06 \) and followed by \( \tau=18.13,27.18,36.27.18,36.28,\ldots \) the spacing of the zeros along the \( \sigma=1 \) line are separated from each other by approximately 9 units. A closeup of the circle contours about the zero at \( s=1+i9.06 \) looks like this-
Another zero of η(s), this time lying along the σ=1/2 axis, is shown on the next contour plot.

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U.H. Kurzweg  
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Gainesville, Florida