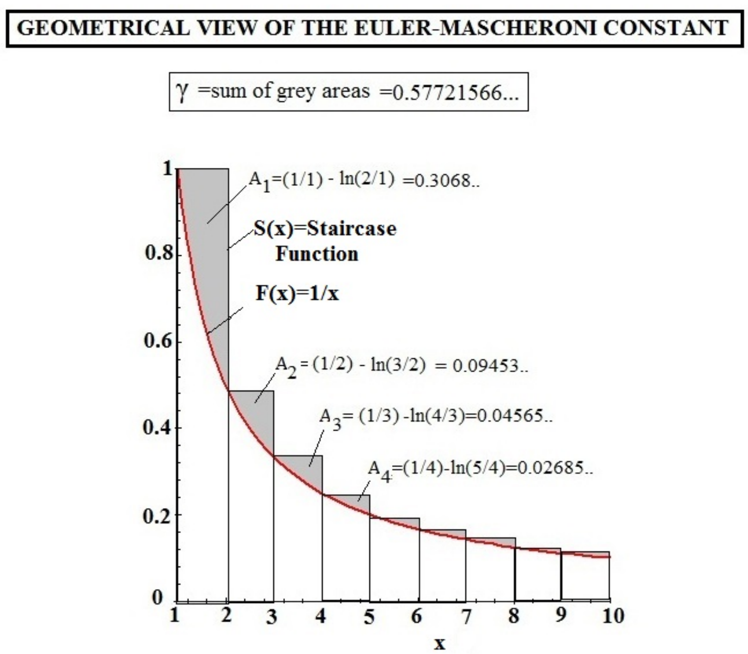
EULER-MASCHERONI CONSTANT

In studying the difference between the divergent area under the curve F(x)=1/x from x=1 to infinity and the area under the staircase function where we have–

$$S(x) = \frac{1}{n} \quad in \quad n \le x \le n+1$$

, the Swiss mathematician Leonard Euler found back in 1734 that the area equals the constant value γ=0.57721566…. The Italian mathematician Lorenzo Mascheroni also studied this constant some fifty years later and thus got his name attached to it. We want here to give the details of its derivation and show how it may be expressed in several different ways.

Our starting point of the discussion is the following graph-



We show there the curve F(x)=1/x plus the extra area (in grey)left over when compared with the step function S(x). Adding up the various increments we get-

A_1=(1/1)-ln(2)+ln(1)

A_2=(1/2)-ln(3)+ln(2)

A_3=(1/3)-ln(4)+ln(3)

so that-

A_n=(1/n)-ln(n+1)-ln(n)

Adding these area increments up out to infinity, we get-

$$A_{total} = \frac{\lim}{n\to\infty}\left\{\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right\}$$

This limit represents the Euler-Mascheroni constant γ. In looking at the sum of the gray areas in the above graph, one sees that its value lies in the range-

$$\sum_{k=0}^{n}\frac{1}{2n(n+1} - \ln(\frac{n+1}{n}) < \gamma < \int_{x=2}^{\infty}(\frac{1}{x} - \frac{1}{x-1})dx$$

So if we take n=100, we get-

$$0.485099.. < \gamma < \ln(2) = 0.693147..$$

Thus the constant γ has a finite value lying somewhere in this range. Such a finite value is interesting since it represents the difference between two infinities.

To find the actual value of this Euler-Mascheroni constant we can try the iteration -

$$f[n+1] = f[n] + \frac{1}{n+1} - \ln(\frac{n+1}{n}) \quad subject \quad to \quad f[1] = 1$$

This yields f[100]=0.5822.. and is thus seen to be a very slowly convergent toward the final goal of γ=0.57721…. To speed things up we can use the following alternate approach. We first define the Psi (digamma) function-

$$\psi(n) = \frac{d}{dx}[\ln(\Gamma(x)]|_{x=n} = \frac{\frac{d}{dx}[\int_{t=0}^{\infty} t^{x-1}\exp(-t)dt]|_{x=n}}{\Gamma(n)} = \frac{\int_{t=0}^{\infty}\ln(t)\,t^{n-1}\exp(-t)dt}{\Gamma(n)}$$

with Γ(x) being the gamma function. From this follows the important integral-

$$\psi(1) = \int_{t=0}^{\infty}\ln(t)\exp(-t)dt = -\gamma$$

This integrand is negative for $0<t<1$ and positive for $1<t<\infty$. This fact suggests we use integration by parts and rewrite things as the sum of two integrals-

$$\gamma = -\int_{t=0}^{1} \ln(t)\exp(-t)dt - \int_{t=1}^{\infty} \frac{\exp(-t)}{t}dt$$

The second integral in this expansion just defines the exponential integral Ei(1)=0.2193839344… .Evaluating both integrals to 50 place accuracy we find that the Euler-Mascheroni constant equals-

γ=0.79659959929705313428367586554252408007320662934683…-
0.21938393439552027367716377546012164903104729340691…

or

γ =0.57721566490153286060651209008240243104215933593992…

This irrational number has in recent years been evaluated to well over twenty billion places.

I remember being first exposed to this constant while finding the Bessel Function of the Second Kind of integer order ν. There one has the Weber indeterminate form-

$$Y_\nu(x) = \frac{1}{\pi}\left\{\frac{\partial J_\nu(x)}{\partial \nu} - (-1)^\nu \frac{\partial J_{-\nu}(x)}{\partial \nu}\right\}$$

with-

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+\nu}}{n!(n+\nu)!}$$

When carrying out this operation one finds a rather messy expansion whose leading term reads-

$$Y_\nu(x) = \frac{2}{\pi} J_\nu(x)[\ln(\frac{x}{2}) + \gamma] + ...$$

clearly showing the presence of the Euler-Mascheroni constant γ.

There are numerous other existing identities involving this constant. Many of these can be found in the Abramowitz and Stegun " Handbook of Mathematical Functions" which you can access at –

http://people.math.sfu.ca/~cbm/aands/toc.htm

In addition the Wikipedia pagehttp://en.wikipedia.org/wiki/Euler-Mascheroni_constant

contains even more of such identities involving γ.

Let us work out a few of these identities. First of all we start with the definition of the Psi(n) function given earlier. This function has the values-

$\psi(1)=-\gamma$, $\psi(2)=1-\gamma$, $\psi(3)=(3/2)-\gamma$, and $\psi(4)=(11/6)-\gamma$

for n=1, 2, 3, and 4. One recognizes that the constants present in these results are just 1+1/2+1/3+1/4+ ..1/n. So we get the identity-

$$\gamma=\sum_{k=1}^{n-1}\frac{1}{k}-\psi(n)$$

This result is not very useful since the sum suffers from the same slow convergence condition as our original definition for γ did. Also we can look at the function–

$$M(z)=[z-\Gamma(\frac{1}{z})]$$

for z=10, 100, 1000, and 10000. This yields-

M(10)=0.4864 , M(100)=0.5674 , M(1000)=0.5762 , M(10000)=0.5771. Thus one can state that-

$$\gamma=\frac{\lim}{z\to\infty}\left\{z-\Gamma(\frac{1}{z})\right\}$$

Again the convergence is seen to be slow requiring a value of z=10,000 to generate a three digit accuracy for γ.

In view of the above results it appears that the best approach for quickly finding γ involves using one form or another of the basic integral definition-

$$\gamma=-\int_{t=0}^{\infty}\ln(t)\exp(-t)dt=\int_{t=0}^{1}\left|\ln(t)\right|\exp(-t)dt-Ei(1)$$

where Ei(1) is the known exponential integral-

$$Ei(1)=\int_{t=1}^{\infty}\frac{\exp(-t)}{t}dt=\frac{\lim}{s\to 1}\{\exp(-1)Laplace(\frac{1}{1+t})\}\ .$$

The other integral may be expanded as-

$$\int_{t=0}^{t=1} \frac{\exp(-t)}{t} dt = \int_{t=0}^{1} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^{n-1} dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{kk!}$$

We thus have the identity-

$$\gamma = -\int_{t=1}^{\infty} \frac{\exp(-t)}{t} dt - \sum_{k=1}^{\infty} \frac{(-1)^k}{kk!}$$

which can quickly be evaluated to any desired order of accuracy and no longer includes a ln(t) term. Some simple variable substitutions and integration by parts also allows us to write-

$$Ei(1) = -\int_{u=0}^{[1/\exp(1)]} \frac{du}{\ln(u)} = \frac{\sum_{k=2}^{n-2} (-1)^k k!}{\exp(1)} + (-1)^{n-1}(n-1)! \int_{t=1}^{\infty} \frac{\exp(-t)}{t^n} dt$$

and-

$$\gamma = -\exp(-1) \int_{t=0}^{1} \frac{dw}{1-\ln(w)} - \sum_{k=1}^{\infty} \frac{(-1)^k}{kk!}$$

The last expression requires an integration over only the limited range 0≤t≤<1. In this last expression we can expand the function 1/[1-ln(w)] in a Taylor series about w=1 and then integrate. This yields-

$$\int_{w=0}^{1} \frac{dw}{1-\ln(w)} = 1 - \sum_{k=1}^{\infty} \frac{(-1)^{k+1} C_k}{k!(k+1)}$$

where, unfortunately, the constants C_k increase rapidly when k gets large making for slow convergence. Using a Pascal-like triangle based on the derivatives of 1/[1-ln(w)] evaluated at w=1, we find the first few of these constants to be-

$$C_1 = 1, C_2 = 1, C_3 = 2, C_4 = 4, C_5 = 14, C_6 = 38, C_7 = 216, and, C_8 = 600 \quad .$$

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