# VARIATIONS ON EULER'S FORMULA

**Some 275 years ago the famous Swiss mathematician Leonhard Euler (1707-1783) discovered the formula-**

$$\exp(i\theta)=\cos(\theta)+i\sin(\theta)$$

**which relates the exponential function with the trigonometric functions , the number π, and the imaginary number i=sqrt(-1) . Many have considered this formula to be the most beautiful in all of mathematics, especially when θ is taken to be π. Its derivation is straight forward using the known infinite series representations for the components. Substituting we have-**

$$\boldsymbol{exp(i\theta)}=1+(i\theta)+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\cdots=\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\cdots\right)+i\left(\frac{\theta}{1!}-\frac{\theta^3}{3!}+\frac{5}{5!}-\cdots\right)$$

$$=\cos(\theta)+i\sin(\theta)$$

**We start with the complex version of his formula, namely,-**

$$\exp(iz)=\cos(z)+i\sin(z) \quad \text{or} \quad \exp(-iz)=\cos(z)-i\sin(z)$$

**Adding together the two exponential terms , we find-**

$$\cos(z)=[\exp(iz)+\exp(-iz)]/2$$

**and the difference yields-**

$$\sin(z)=[\exp(iz)-\exp(-iz)]/(2i)$$

**On seting x=0 in z=x+iy, these last two equations produce the hyperbolic functions-**

$$\cos(iy)=[\exp(+y)+\exp(-y)]/2=\cosh(y)$$

**and-**

$$\sin(iy)=[\exp(-y)-\exp(y)]/(2i)=i\sinh(y)$$

**Also from the Pythagorean Theorem we have that sin(iy)^2+cos(iy)^2=1. So it must be true that-**

$$\cosh(z)^2-\sinh(z)^2=1$$

**This ability to express sine and cosine in terms of exponentials allows one to quickly establish numerous other identities. We have , for example, that-**

$$I=\int_{n=0}^{\infty}\cos(x)\exp(-x)dx=1/2$$

**One can write this as-**

I=Re{$\int_{n=0}^{\infty} exp[(i-1)x]dx$}=Re{$\frac{exp[(i-1)x]}{(i-1)}$=Re((1+i)/2]=1/2

There are other ways to get this result such as integration by parts or Laplace transforms with s=1. However none are as elegant and simple as the use of Euler`s Formula. One can also carry out differentiations of products involving sin(x) and cos(x). Consider the derivative -

d[sin(x)cos(x)]/dx=(1/2)d[sin(2x)]/dx

In terms of exponentials we find the simple result-

(1/2)Im{exp(2ix)/i}=-cos(2x)

We point out that Euler`s Formula, expressed as cos(iy) also yields the alternate correct form-

exp(y)=exp(-y)+2cos(iy)

Upon setting y =π, we get the formula-

G-(1/G)=2cos(iπ)

, where

G=exp(π)= 23.140692632779269005729086367948547380266106242600 .

This number was proven to be transcendental by the mathematician Gelfond in the early 1930s. There are an infinite number of other constants which follow from the Euler Formula. For example-

N=exp(aπ)

, where 'a' is any number, rational or irrational. This number satisfies the equality-

N+ (1/N)=2cosh($\pi a$)

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