PROPERTIES OF EXP(Z) FOR COMPLEX Z=X+iY

One learns in beginning calculus classes that there exists an important complex function defined as-

$$F(z)= exp(z) = \frac{lim}{m\to 0}[1+zm]^{\wedge}(1/m) \text{ , where } z=x+iy .$$

On applying the binomial theorem to F(z) this function expands to-

$$F(z)=1+(z)^{\wedge}1/1!+(z)^{\wedge}2/2!+(z^{\wedge}3)/3!+\ldots= \sum_{k=0}^{\infty}\frac{z^{\wedge}}{k!}$$

It is the purpose of this note to look at the properties of this function in greater detail.

EXP(1):

Let us begin with z=1 where F(z) converts to the irrational number e . My computer gives the first one hundred digits for this number as-

e=exp(1)= 2.718281828459045235360287471352662497757247093699959574966967627724076630353547594571382178 5251664274

Several years ago I constructed a mnemonic for 'e' which goes as follows-

2.7-twice Andrew Jackson inauguration (1828,1828)-right triangle(45,90,45)-first three primes(2,3,5)-full circle(360)deg)-year before crash(28)-early Boing jet(747)-end of plaque in Europe(1352)-famous auto route west(66)

This gives exp(1) to 32 place accuracy. Because of the factorials in the denominator of the e expansion, there is no major problem in obtaining the accuracy for e to well over a billion places. The fact that the number is irrational means that the e expansion does not have any recurring identical packets. One can telescope the e series . Taking two terms at a time produces the more rapidly converging series-

$$e= \sum_{k=1}^{\infty}[\frac{2n}{(2n-1)!}] = 2/1! + 4/3!+ 6/5! + 8/7!+\ldots$$

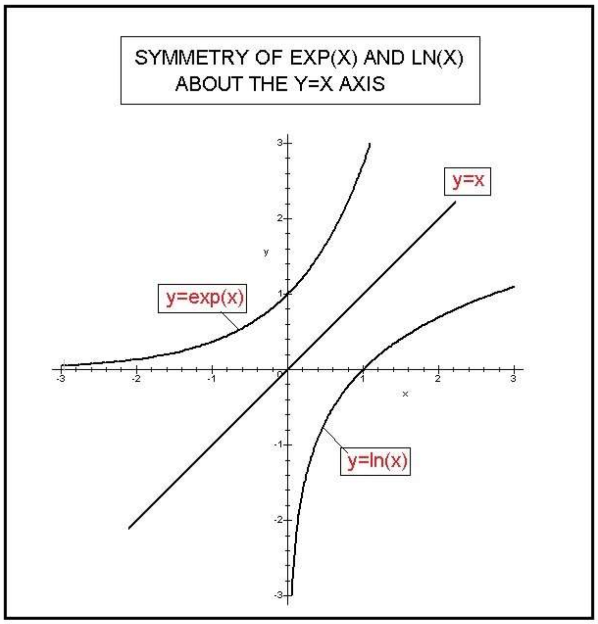
Several years ago we found a new use for the irrational number e. It turns out that chunks of this number when very small numbers are added can produce large prime numbers. Here is such an example constructed by a fifty digit chunk of e with the small number 123 added-

P=7182818284590452353602874713526624977572470937123

By using parts of the products of several different irrational numbers one has a new way to transmit openly large primes in coded form. This should allow some new approaches for people working in cybersecurity.

EXP(X):

This special case of exp(z) defines a function which goes to zero as x becomes minus infinity, has value one when x equals zero and goes to infinity as x goes toward plus infinity. Here is a plot of this function together with its inverse, the natural logarithm of x –



, The area under the exp(x) curve extending from –a to +a equals-

Area=$\int_{t=-a}^{+a} exp(t)dt =$ exp(a)-exp(-a)

The finite area extending from x= - ∞ to x=0 is precisely one. Note that the derivative of exp(x) is equal to exp(x). Hence the differential equation for exp(x) is just-

dy/dx=y subject to y(0)=1

EXP(iY):

For this form we get the expansion-

exp(iy)=1+(iy)+(iy)^2/2!+(iy)^3/3!+(iy)^4/4!+….

Regrouping and using the correct form for the various powers of i, we get-exp(iy)=[1-y^2/2!+y^4/4!+]+i[y-y^3/3!+y^5/5!+]= cos(y)+isin(y)

This result shows that-

cos(θ)=Re[exp(iθ)] and sin(θ)=Im[exp(iθ)]

Using this information we can quickly solve –

$$\int_{\theta=0}^{\pi/2} \boldsymbol{sin(2\theta)\ d\theta = Im\{\int_{\theta=0}^{\frac{\pi}{2}} exp(2i\theta)d\theta\} = Im\{[exp(i\pi)-1]/(2i)\} = 1}$$

Notice that if we set y to π, we get-

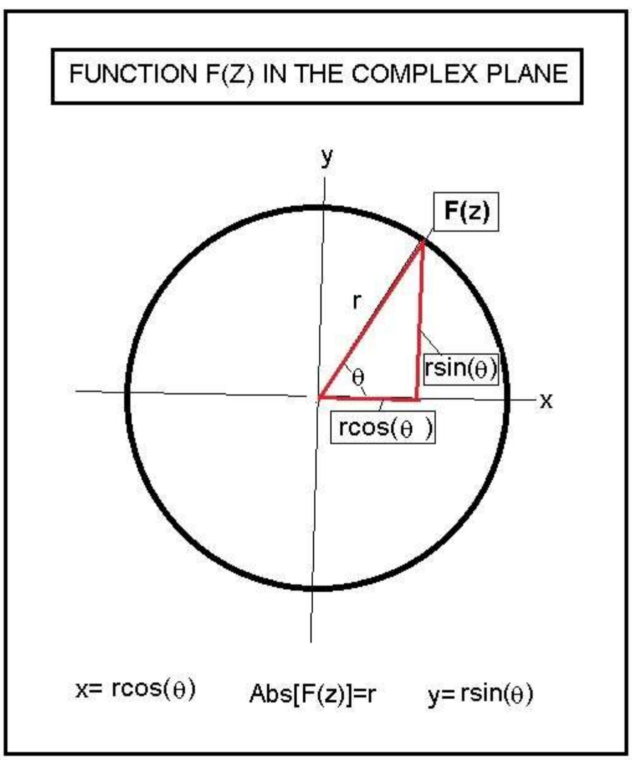
$$e^{i\pi}+1 \ \ =0$$

This result, due to Leonard Euler , is considered by many to be the most beautiful formula in all of mathematics since it links together the fundamental constants e, π, i, 1, and 0.

EXP(X+iY):

Finally we come to the most general form of the function F(z). To recognize its components in the z plane we first write out its polar form-

exp(x+iy)=rexp(iθ)=r[cos(θ)+i sin(θ)]o F(z) represents a circle of radius r in the z plane as shown-



Note that the red triangle dimensions satisfy the Pythagorean theorem x^2+y^2=1. If we set r=1 then F(x+iy) has value F(1,0)=1, F[0,1]=I^1, F[-1,0]= i^2, and F(0,-1)= i^3. When r is a fun ction of angle one gets a spiral.

One can also sum up and subtract F(z) and F(-z) to generate the hyperbolic functions-

cosh(x)= [F(x)+ F(x)]/2 and sinh(x)= [F(x)-F(x]/2i

, provided y is set to zero. Here cosh(x) is an even function and sinh(x) an odd function. They combine as-

cosh(x)^2-sinh(x)^2=1 .

U.H.Kurzweg
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Gainesvile, Florida