SOME LESS WELL KNOWN IDENTITIES INVOLVING N!

It is well known that the factorial is defined as-

$$n!=1\cdot 2\cdot 3\cdot 4\cdot \ldots \cdot n=\int_{t=0}^{\infty} t^n exp(-t)dt=\Gamma(n+1)$$

, where Γ(n) is the continuous gamma function. We also have the identities-

$$n!(n+1)=(n+1)! \quad \text{and} \quad n\Gamma(n)=\Gamma(n+1)$$

Using this information, one can construct numerous other identities involving the factorial. Doing so will be the main topic of this article.

We begin with writing (2n)! as-

$$(2n)!=[1\cdot 3\cdot 5\cdot \ldots\cdot(2n-1)]\cdot[(2\text{^}n)\cdot n!]$$

Next we notice that Γ(1/2)=sqrt(π), Γ(3/2)=(1/2)sqrt(π), and Γ(5/2)=(3/2)(1/2)sqrt(π). These allow us to generalize to produce-

$$[1\cdot 3\cdot 5\cdot \ldots\cdot(2n-1)]=(2\text{^}n)\Gamma(2n+1)/sqrt(\pi)$$

Getting rid of the left hand side of this last equality, allows us to rewrite (2n)! as-

$$(2n)!=n!\{2\text{^}(2n)\cdot\Gamma[(2n+1)/2]/sqrt(\pi)\}$$

This result was first obtained by Legendre and is known in the literature as Legendre's duplication formula. At n=5, it reads-

$$10!=5!2^{10}\Gamma(11/2)/sqrt(\pi)=3628800$$

We can double (2n)! by replacing n by 2n. This produces-

$$(4n)!=n!2\text{^}(6n)\{\Gamma[(4n+1)/2]\Gamma[(2n+1)/2]\}/\pi)\}$$

Doubling the ns again produces-

$$(8n)!=n!2\text{^}14n\{\Gamma[(8n+1)/2]\Gamma[(4n+1)/2]\Gamma[(2n+1)/2]\}/[\pi sqrt(\pi)]$$

Also we have-

$$(16n)!=n!2\text{^}30n\{\Gamma[(16+1)/2]\Gamma[(8n+1)/2]\Gamma[(4n+1)/2]\Gamma[(2n+1)/2]\}/\pi\text{^}2$$

So we have the generalization-

$$[(2^k)n]!=\{n!2^{[2n(2^k-1)]}/(\mathrm{sqrt}(\pi)^k\}\prod_{j=0}^{k-1}\Gamma[\frac{2n(k-j)+1}{2}]$$

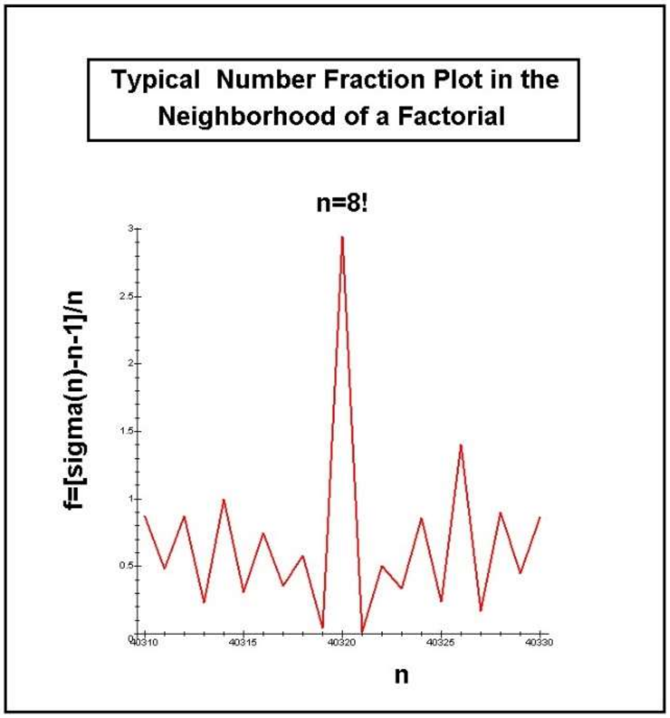
For k=1 this result reduces back to the Legendre duplication formula. This generalization allows one to easily express large n! in terms of products of powers of primes.Take the case of k=2 and n=2. Here we get-

8!=(2^7)(3^2)*5*7=40320 with the exponent vector V= [7 2 1 1]

Note that this factorial and others have a similar prime product form in which the lowest primes have the largest exponents. Here is a demonstration of this fact via the following table-

Factorial	exponent vector, V
2!=2	[1 0 0 0]
3!=6	[1 1 0 0]
4!=24	[3 1 0 0]
5!=120	[3 1 1 0]
6!=720	[4 2 1 0]
7!=5040	[4 2 1 1]
8!=40320	[7 2 1 1]
9!=362880	[7 4 1 1]
10!=3628800	[8 4 2 1]

Numbers of this type, where the powers of their prime vector V elements drop with increasing prime number such as 8>4>2>1>0 for 10!, are what we have termed earlier as super-composites. Such super-composites are recognized by having their number fraction f=[sigma(n)-n-1] /n >1. The values of the number fraction are found to tower in value above their neighbors as shown in the following graph for f(8!)-



One can also use ever increasing n! to show that $f(\infty)$ becomes unbounded but does so very slowly. Even for f(100!) we find the relatively small value $f=7.293771353\ldots$. Remember that when n is a prime, the value of f will always be zero.

We can quickly determine 11! by adding up the vectors [8 4 2 1] and [0 0 0 0 1].That is, 11! corresponds to the vector[8 4 2 1 1]. So-

$$11!=(2^8)(3^4)(5^2)(7^1)(11^1)=39916800 \quad \text{with} \quad f(11!)=3.609814790\ldots$$

Another interesting relation involving n! and not often seen in the literature is the fact that-

$$1\cdot 4\cdot 9\cdot 16\cdot \ldots . n^2=n!n!$$

That is, the product of the squares of integers up to n^2 equals the product of $n!\cdot n!$ One can take this result further by noting that-

$$1\cdot 2^m\cdot 3^m\cdot 4^m= (n!)^m$$

Thus we have that-

$$1\cdot 16\cdot 81\cdot 256 = (4!)^4 = 331726$$

One can rewrite these equalities in the compact form-

$$\prod_{j=1}^{n} j^m = (n!)\hat{}m$$

So, 1·8·27·64=13824 and 1·8·27·64·125·256=(6!)^3=373248000.

Finally we can look at the sum of some series involving n! Lets begin with the sum-

G(n)=1!+2!+3!+4!+(n-1)!+n!

Here the sums go as G(1)=1, G(2)=3, G(3)=9, G(4)=33, and G(5)=158. Summing to n! we get-

$$G(n)=\sum_{j=1}^{n} j!$$

This is a very rapidly growing series as n increases. Thus, for example ,

G(20)= 2561327494111820 3

Next we look at the numbers –

$$N = 1+1+1/6+1/24+1/120+\ldots = \sum_{n=0}^{\infty}\frac{1}{n!} = e = 2.71828182845235\ldots$$

and-

$$M = 1+1+1/36+1/76+1/14400+\ldots = \sum_{n=0}^{\infty}\frac{1}{n!n!} = I_0(2) = 2.279585302\ldots$$

, where $I_0(2)$ is the modified Bessel function of order zero at x=2. We also have

$$\sum_{n=0}^{\infty}\frac{1}{n!n!n!} = 2.12970254898330641 81\ldots$$

and-

$$\sum_{n=0}^{\infty}\frac{1}{n!n!n!n!} = 2.06327462384631523 14\ldots$$

Clearly as the number of n!s increase in the denominator of the sum, the series heads toward two.

U.H.Kurzweg
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Gainesville, Florida