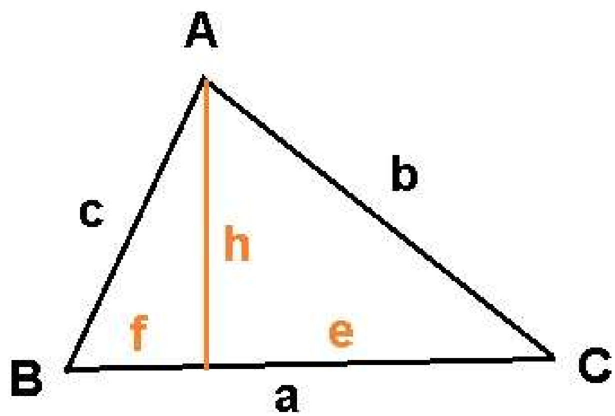
FOUR BASIC LAWS OF TRIGONOMETRY

I is well known that any triangle (right or not) obeys the four laws of Sin, Cos,Tan and Mollweide. We give here a quick derivation of these formulas. Our starting point is a oblique triangle of side lengths a, b, and c and opposite angles A, B, and C measured in degrees. A red bisector of length h coming from vertex A is also added as indicated in the following picture-



From the figure one has at once that-

$$b\sin(C)=c\sin(B)=h$$

From this folows the Law of Sines-

$$\frac{a}{\sin(A)}=\frac{b}{\sin(B)}=\frac{c}{\sin(C)}$$

Next we look at cos(C)= e/b with e+f=a, h^2 =b^2-e^2=c^2-f^2. Eliminating e,f, and h, we have the Law of Cosines-

$$c^2=a^2+b^2-2ab\cos(C)$$

To get the forms for the remainng, less used trig formulas, we start with the first two terms in the Law of Sines to get-

$$a/\sin(A)=b/\sin(B)=k$$

So-

(a-b)/(a+b)=[sin(A)-sin(B)]/[sin(A)+sin(B)] =tan[(A-B)/2] /tan[(A+B)/2]

To get this result we have set A=U+V and B=V-U. The resultant calculation yields the Tangent Formula-

$$\frac{(a-b)}{(a+b)}=\frac{\tan[\frac{(A-B\}}{2}]}{\tan[\frac{(A+B)}{2}]}$$

Finally we consider the derivation of the Mollweide Formula starting with-

$$\frac{a+b}{c}=\frac{\sin(A)+\sin(B)}{\sin(C)}$$

, since the constant k cancels out. Again letting A=U+V and B=V-U we get-

$$\frac{a+b}{c}=\frac{2\sin(V)\cos(U)}{\sin(C)}=\frac{2\sin[(A+B)/2]\cos[(A-B)/2]}{\sin(C)}$$

One last expansion, using A+B+C=π, produces the Mollweide result-

$$\frac{a+b}{c}=\frac{\cos[\frac{A-B}{2}]}{\sin(\frac{C}{2})}$$

It is interesting that this last trigonometric formula contains the length of all sides of the triangle plus its three vertex angles.

Let us apply the above four formulas to a particular triangle of sides a=2, b=3 and a vertex angle between these two sides given as C=120 deg.. From the Law of Cosines this says-

c=sqrt[4+9-2(6)cos(120)]=sqrt(19)= 4.35889…

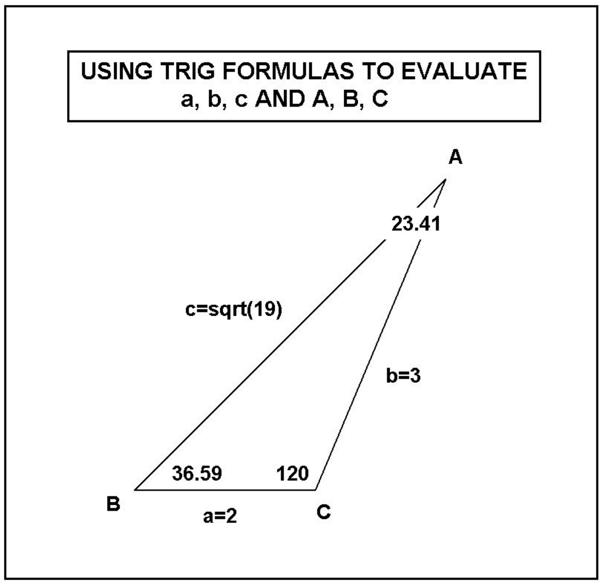
Next the Law of Sines yields-

A=arcsin[2*sin(120)/sqrt(19)] =23.413224

and

B=arcsin [3sin(120)/sqrt(19)]= 36.586775

As expected the three angles A, B, and C add up to 180 deg. We now have found all side-lengths and vertex angles. A scaled picture of the triangle follows with vertexes at [0,0],[2,0], and [sqrt(19)cos(B),3sin(60)]-



Note that the Mollwiede Formula is also satisfied as indicted-

(2+3)/sqrt(19)=1.147078= cos(6.586773)/0.866025

So is the Tangent Law-

(2-3)/5=tan[(A-B)/2]/tan[(A+B)/2]=-0.11547/0.57733= -0.2000… .

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