FUNCTION APPROXIMATION USING INTEGRALS CONTAINING EVEN LEGENDRE POLYNOMIALS

About fifteen years ago (https://mae.ufl.edu/~uhk/EVAL-ARCTAN.pdf) we came up with a new method for approximating the arctan function using Legendre polynomials. The method now referred to as the KTL technique was later extended by us to all trigonometric functions (https://mae.ufl.edu/~uhk/KTL-METHOD.pdf) and should be applicable for approximating many other slowly varying functions provided the Legendre polynomials used in the technique vary rapidly and have even symmetry in

[-1,1]. It is our purpose here to generalize the KTL method making it applicable for any integral of the form-

$$J(n,a)= \int_{x=0}^{1} P[2n,x)f(ax)dx = const.\{h(a)N(n,a) - g(a)M(n,a)\}$$

, with g(a) and h(a) related back to f(ax).The N(n,a) and M(n,a) are polynomials in 'a' once the value of n used in the Legendre Polynomial is specified. They increase in size as n gets larger. The polynomials N and M are easiest to obtain via the MAPLE computer operation -

collect(J(n,a),{h(a),g(a)})

Here P[2n,x] represents the rapidly oscillating even Legendre Polynomials with n zeros in [0,1] . The essence of the KTL method is that the integral J(n,a) approaches zero value as n goes to infinity. So we have the KTL approximation-

$$g(a)/h(a) \approx N(n,a)/M(n,a)$$

, with n chosen beforehand and typically set at n=4 or above. We first came up with this approximation when playing around with the even Legendre Polynomial P(2,x) =(3x^2-1)/2 and f(1,x), =1/(1+x^2) back in 2009. This produced –

$$J(1,1)=\int_{x=0}^{1} \frac{P(2,x)}{(1+x^2)dx} = (3-\pi)/2=0.07796$$

On setting this value to zero, one obtains the π approximation of 3. This result suggested to me to let n become larger in order to improve the approximation for π.

Indeed, going to 2n=10 produces-

$$J(5,1)=(269852/315)-\pi(4363/16)=0.0000309$$

It leads to the improved approximation-

$$\pi \approx 4317632/1374345=3.141592\ldots$$

good to six places. Clearly the approximation improves rapidly with increasing n. Note here that n must be even to get the π approximation. For odd n one gets a poorer approximation for ln(2) . Also the replacement of the Legendre Polynomials by the Chebyshev Polynomials in the method also works but yields a poorer approximation for the same n.

To generalize things we replace f(ax)=1/(a^2+x^2) as used in the arctan approximation and f(ax)=cos(ax) as used for the tan(a) approximation by any other symmetric functions of f(ax) having no poles in [0,1].

We have the more general starting formula-

$$J(n,a)=\int_{x=0}^{1} P(2n,x)f(ax)dx$$

, where P(2n,x) is the 2nth Legendre Polynomial and f(ax) is any function with an even symmetry in [-1,1] and which varies only slightly over the range [0,1]. One can use one of the following larger 2n Legendre Polynomials in the present analysis-

$$P_0 := 1$$

$$P_2 := -\frac{1}{2} + \frac{3}{2}x^2$$

$$P_4 := \frac{3}{8} - \frac{15}{4}x^2 + \frac{35}{8}x^4$$

$$P_6 := -\frac{5}{16} + \frac{105}{16}x^2 - \frac{315}{16}x^4 + \frac{231}{16}x^6$$

$$P_8 := \frac{35}{128} - \frac{315}{32}x^2 + \frac{3465}{64}x^4 - \frac{3003}{32}x^6 + \frac{6435}{128}x^8$$

$$P_{10} := -\frac{63}{256} + \frac{3465}{256}x^2 - \frac{15015}{128}x^4 + \frac{45045}{128}x^6 - \frac{109395}{256}x^8 + \frac{46189}{256}x^{10}$$

$$P_{12} := \frac{231}{1024} - \frac{9009}{512}x^2 + \frac{225225}{1024}x^4 - \frac{255255}{256}x^6 + \frac{2078505}{1024}x^8 - \frac{969969}{512}x^{10} + \frac{676039}{1024}x^{12}$$

The larger n is taken the smaller the integral J(n,a) will become and the better the approximation representing g(a)/h(a) becomes. Setting J(n,a) to zero then gives us the generalized KTL approximation-

g(a)/h(a)=N(n,a)/M(n,a)

for different functions f(ax). Here is a brief list of many functions g(a)/h(a) which can be approximated by the indicated symmetric f(ax) -

f(ax)	g(a)/h(a)
1/(a^2+x^2)	(1/a)arctan(1/a)
cos(ax)	tan(a)
exp-(ax)^2	exp(a) erf(sqrt(a))
1/(1+(ax)^2)	arctan(sqrt(a))
cosh(ax)	tanh(a)

Let us demonstrate in more detail the approximation method when using p(ax)=cosh(ax). We start, after choosing n=2, with the integral-

$$J(2,a)=\int_{x=0}^{1} P(4,x)\cosh(ax)dx$$

It evaluates to-

sinh(a) [a^4+45a^2+105]-cosh(a) [10a^3+105a]

after setting J(2,a) to zero. The final result is –

tanh(a)≈(10a^3+105a)/(a^4+45a^2+105)

This result is surprisingly close to the exact value. At a=1 the difference between the exact value of tanh(1) and the approximation 115/151=0.761589 is only about 0.00001. It will become even closer if n is increased from 2 or 'a' is decreased to less than one. Notice that this approximation for tanh(a) is all there is needed to get approximations for other hyperbolic functions. We have the identities-

sinh(x)=tanh(x)/sqrt(1-tanh^2(x)) and cosh(x)=1/sqrt(1-tanh^2(x))

So using the above tanh approximation we get -

$$\sinh(1)\approx \frac{115}{sqrt(151^2-115^2)} = 1.17518 \quad \cosh(1) \approx \frac{1}{sqrt(151^2-115^2)} = 1.54306$$

These results are good to at least three decimal places.

We have shown in the above discussion that the KTL approximation method can be extended to multiple other functions g(a) and h(a) not already used earlier to obtain our arctan and trigonometric approximations. The accuracy of the method increases with the use of larger even Legendre Polynomials appearing as a product with an even function p(ax) in an integral extending over the finite range [0,1].

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