FUNCTIONAL EQUATIONS WITH INTEGER SOLUTIONS

A special subclass of functional equations are those whose solutions form sequences of integers. One of the best known of these equations is-

$$f(n+2)=f(n+1)+f(n) \text{ subject to } f(1)=f(2)=1$$

A simple substitution produces f(3)=2, f(4)=3, f(5)=5, f(6)=8, f(7)=13, etc. So we find the integer point solution-

$$f(n)=\{1,1,2,3,5,8,13,\ldots)$$

This is the famous Fibonacci Sequence first found by him in 1202. It simply says that the n+2 term equals the sum of the n+1 term plus the nth term. Binet gave a closed form solution for every element in this sequence. It reads-

f(n)=[1/sqrt(5)]{((1+sqrt(5))/2)^n –((1-sqrt(5))/2)^n}

At n=20, we get f(20)=6765. It is amazing that such a complicated expressions involving the irrational sqrt(5) leads to integer values.

It is our purpose here to find and discuss other functional equations which produce integer solutions.

Let us begin with-

$$f(n+1)=(n+1)\, f(n) \text{ subject to } f(1)=1$$

Here we find f(2)=2, f(3)=6, f(4)=24,…

From these integer results we have at once that-

$$f(n)=n! \quad \text{and} \quad (n+1)!=(n+1)n!$$

Consider next the functional equation-

$$f(n+1)=f(n)+(n+1) \quad \text{subject to } f(1)=1$$

Here an evaluation produces-

$$f(n)=\{1,3,6,10,15,21,\ldots\}$$

So the individual elements are given by-

$$f(n)=n(n+1)/2=\sum_{k=1}^{n} k$$

which represents the sum of the integers up through k=n. Making a small variation yields-

f(n+1)=f(n)+(n+1)^2 subject to f(1)=1

This produces the solution sequence-

f(n)={1,5,14,30,55,91,…}

One recognizes the elements as the sum of the squares of the integers-

$f(n)=\sum_{k=1}^{n} k^2=n(n+1)(2n+1)/6$

Likewise, the functional equation-

f(n+1)=f(n)+(n+1)^m subject to f(1)=1

has the solution –

$f(n)=\sum_{k=1}^{n} k^m$

provided m is a positive integer.

Another functional equation with integer solutions is-

f(n+1)= f(n)+2n+1 subject to f(1)=1

It solves as –

f(n)=n^2={1,4,9,16,25,36,…}

Another functional equation with integer answers is-

f(n+1)=(-1)^nf(n)+n subject to f(1)=1

The one line computer program which finds f(n) for n=1 to 20 is here-

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f[1]:=1;  for n from 1 to 20 do f[n]:=(-1)^n*f[n]+n od;
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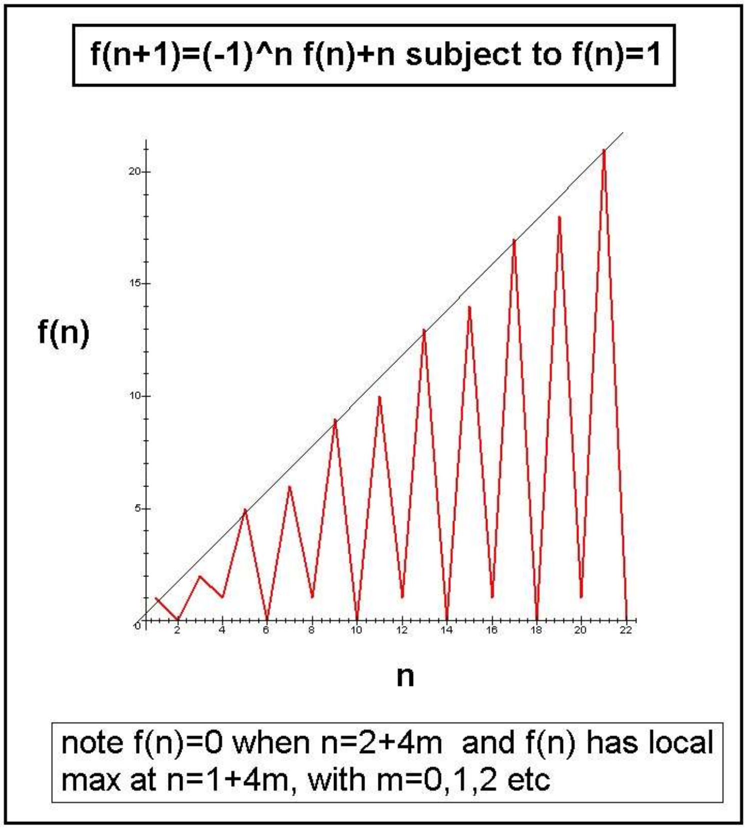
It produces the graph-



Notice that, because of the (-1)^n term in this equation, the solution f(n) does not grow very rapidly and also has zero values at integers n=2+4m.

As one final functional equation with integer answers consider-

f(n+2)=(n+1)f(n) subject to f(1)=f(2)=1

Here we find f(3)=2, f(4)=6, f(5)=24 and f(6)=120. This means we have as an integer solution-

f(n)=Γ(n), with the gamma function satisfying Γ(n+2)=(n+1)$\Gamma(n)$

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