PROPERTEES OF THE GAMMA FUNCTION

The gamma function is defined by the integral-

$$\Gamma(z) = \int_{t=0}^{\infty} t\hat{}(z-1)\,\exp(-\mathrm{t})\,dt$$

, with Γ(1)=1 and z=x+iy. The easiest way to verify that this definition must be true is to note that the Laplace transform of t^(z-1), when s is set to one, just equals Γ(z) . The value for z=2 and z=3 follow from integration by parts and yield-

Γ (2)=1 and Γ(3)=2

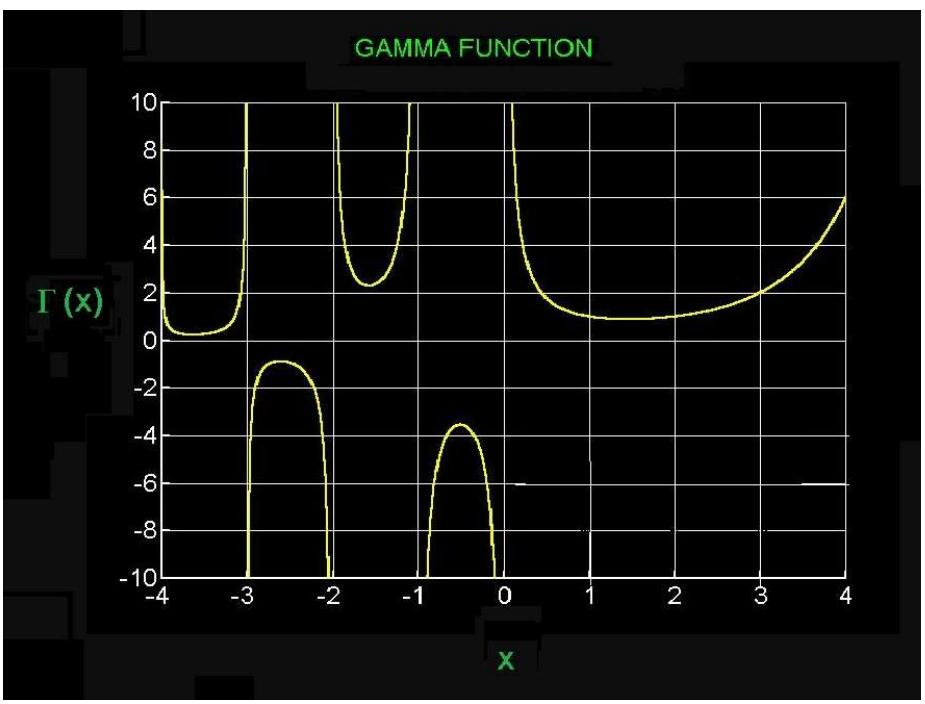
More generally, starting with Γ(z+1), we have –

$$\Gamma(z+1) = \int_{t=0}^{\infty} t^z \exp(-t)\,dt = z\,\Gamma(z)$$

The last equality, again obtained by an integeration by parts, can be considered a generating function valid for any z=x+iy. We have, for example, that-

$$\Gamma\left(\frac{1}{2}\right) = \int_{t=0}^{\infty)} \frac{\exp(-t)}{sqrt(t)}\mathrm{dt} = 2\int_{u=0}^{\infty} \exp(-u^2)\,d\,\mathrm{u} = \mathrm{sqrt}(\pi)$$

. A gaph for the special case where z=x follows-



Notice that the gamma function has infinite values at all negative integers and also at z=x=0. Math tables for Γ(x) typically extend only between x=1 and x=2.This stems from the fact that the above generating function covers the values of Γ(x) at all other xs once the exact values for Γ(x) in 1<x<2 are known.

There are numerous other identities which follow from the above basic integral definition for Γ(z). One of the more useful of these is-

$$\Gamma(z)\ \Gamma(z)=\pi/\sin(\pi z)$$

Its proof is rather lengthy but its validity is easily verified by the following table-

z	Γ(z)	$\Gamma(1-z)$	Γ(z)*Γ(1-z)	π/sin(πz)
1/4	3.6256099	1.2254167	4.442882938	4.442882938
1/2	1,7724538	1.7724538	3.141592654	3.141592654
3/4	1.2254167	3.6256099	4.442882938	4.442882938

Here the integral for Γ(1-z) reads-

$$\Gamma(1\text{-}z)=\int_{t=0}^{\infty}\exp(-t)\,/t^{z}dz=\lim_{s\to 1}\text{Laplace[t^(-z)]}$$

Another interesting identity, this time linking the gamma function with the beta function, is-

$$B(n,m)=\int_{t=0}^{1}t^{n-1}(1-t)^{m-1}dt=\frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}$$

I remember using this identity extensively some sixty years ago while working on my PhD dissertation at Princeton University. Again the derivation is rather lengthy, so we again verify things by simply looking at a couple of examples without further proof. We look at--

$$B(4,3)=\int_{t=0}^{1}t^{3}(1-t)^{2}dt=\frac{\Gamma(4)\Gamma(3)}{\Gamma(7)}=\frac{3!2!}{6!}=\frac{1}{60}$$

and-

$$B(8,3)=\int_{0}^{1}t^{7}(1-t)\text{^}2dt=\frac{7!2!}{10!}=\frac{1}{360}$$

Both integrals agree with the equivalent gamma function quotients.

Another function directly expressible in terms of the gamma function is the Digamma Function . It is defined as-

$$\Psi(z)=[d\Gamma(z)/dz]/\Gamma(z)=d[\ln(\Gamma(z)]/dz$$

Expressed as the quotient of two integrals it reads-

$$\Psi(z)=\frac{\int_{t=0}^{\infty}t^{z-1}\exp(-t)\ln(t)dt}{\int_{t=0}^{\infty}t^{z-1}\exp(-t)dt}$$

Some values are-

$\psi(1)=-\gamma$, $\psi(2)=1-\gamma$, $\psi(3)=3/2-\gamma$, $\psi(4)=\frac{11}{6}-\gamma$

with $\gamma=0.57721566\ldots$ being the Euler-Mascheroni constant. Generalizing things we have-

$$\psi(z)=-\gamma+\sum_{n=1}^{(z-1)}\frac{1}{n}$$

Thus –

$$\text{Psi}(5)=-\gamma+\{1+1/2+1/3+1/4\}=-\gamma+25/12$$

One can also deduce the value of an incomplete partial harmonic series going from n=1 to (z-1) by $\psi(z)+\gamma$. Taking z=101, we find-

$$\sum_{n=1}^{100}\frac{1}{n}=\gamma+\psi(101)=5.1873775\ldots$$

Although this value is still finite, it becomes unbounded as z approaches infinity just as we learned back in our introductory calculus course many years ago.

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