GENERATION OF CLASSIC 2D CURVES USING FIRST ORDER DIFFERENTIAL EQUATIONS

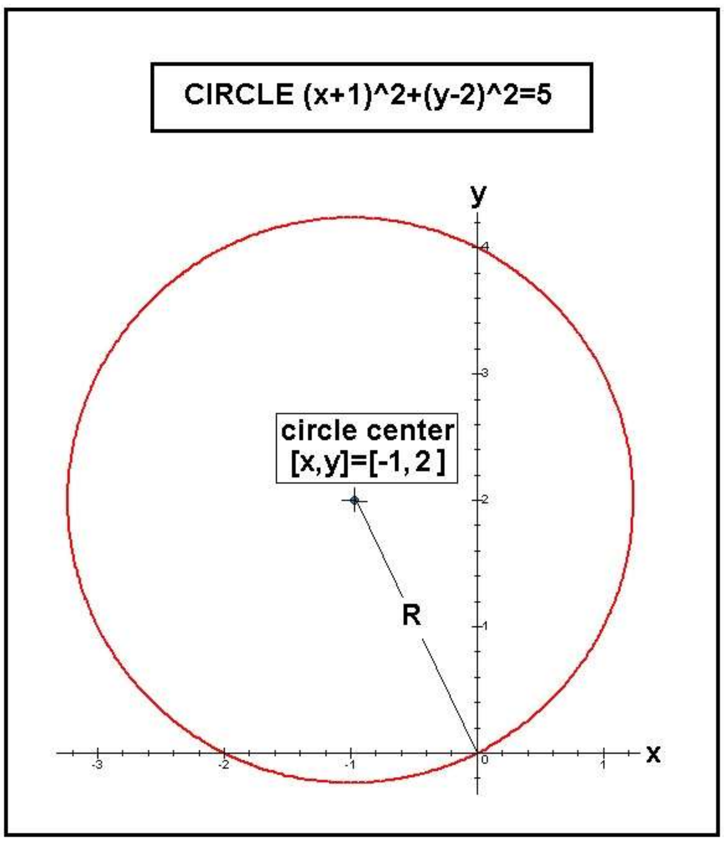
All classic 2D curves can be generated by their own first order differential equation. A simple example is the equation-

dy/dx=-(x-a)/y-b)

It solves as-

(x-a)^2+(y-b)^2=R^2 .

We recognize this to be a circle centered at x=a and y=b with radius R. Here is its graph for a=-1,y=2, and R=sqrt(5)-



The appropriate first order differential equation for any 2D curve is given by-

dy/dx=F(x,y) in Cartesian form or dr/(rdθ)=G(r,θ) in polar form

Note that both dy/dx and dr/(rθ) represent the tangent of θ.

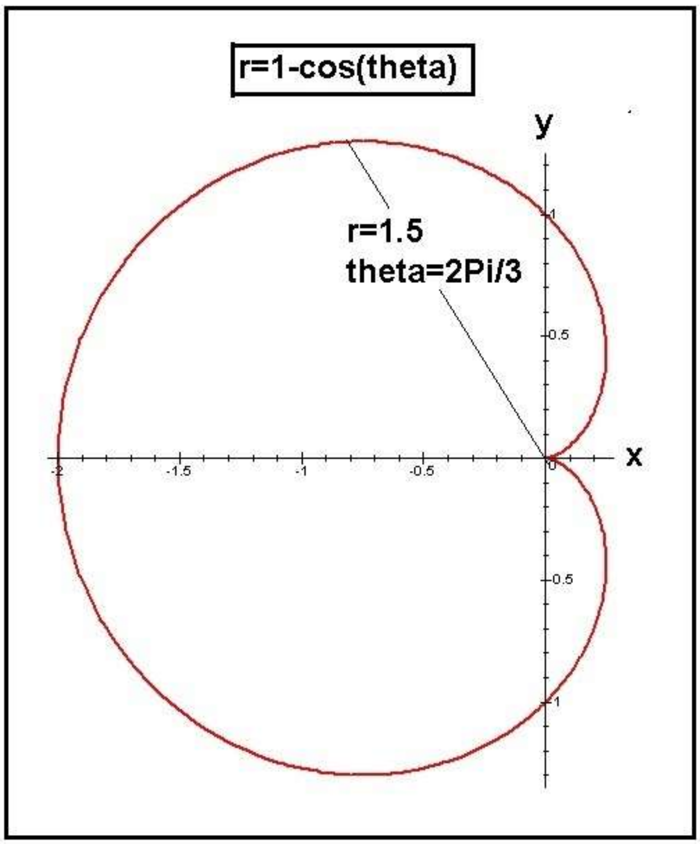
Let us now proceed with generating a few classic curves. Let us begin with-

$$dr/(rd\theta)=\sin(\theta)/r=\tan(\theta)$$

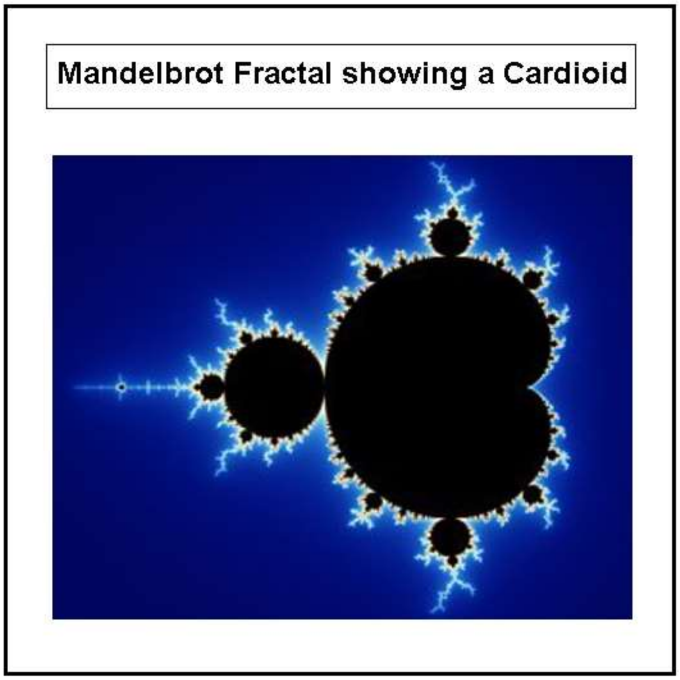
This solves as –

$$r=1-\cos(\theta)$$

provided we impose the condition that r=0 when θ=0. Here is its plot-



One recognizes this to be a Cardioid. It can also be constructed by following a point A on the periphery of a unit radius circle rolling about a stationary circle of the same radius. The cardioid appears in several areas including the caustic formed at the bottom of a tea cup and in connection with the Mandelbrot fractal shown-



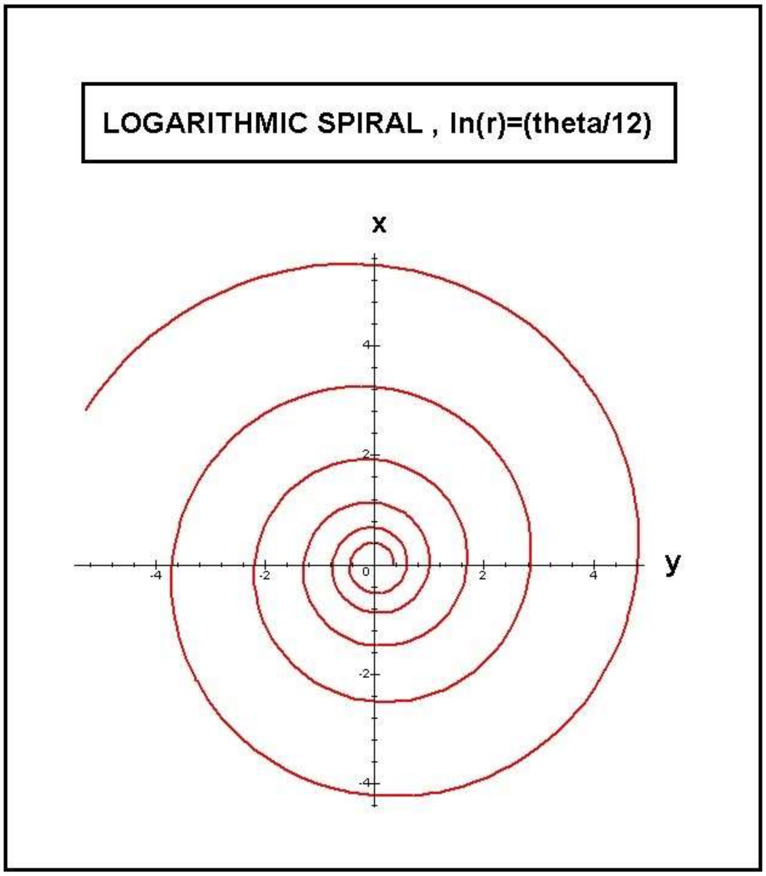
One can also express the cardioid in terms of the more complicated Cartesian form-

$$(x^2+y^2+x)^2=x^2+y^2$$

The next differential equation to consider is-

$$dr/(rd\theta)=1/12$$

Integrating this produces the logarithmic spiral $\ln(r)=\theta/12$ off Bernoulli shown-



, provided r=1 when θ=0. Jacques Bernoulli (1654-1705) was so proud of his discovery of this spiral that he had it engraved on his tombstone in Basel Switzerland.

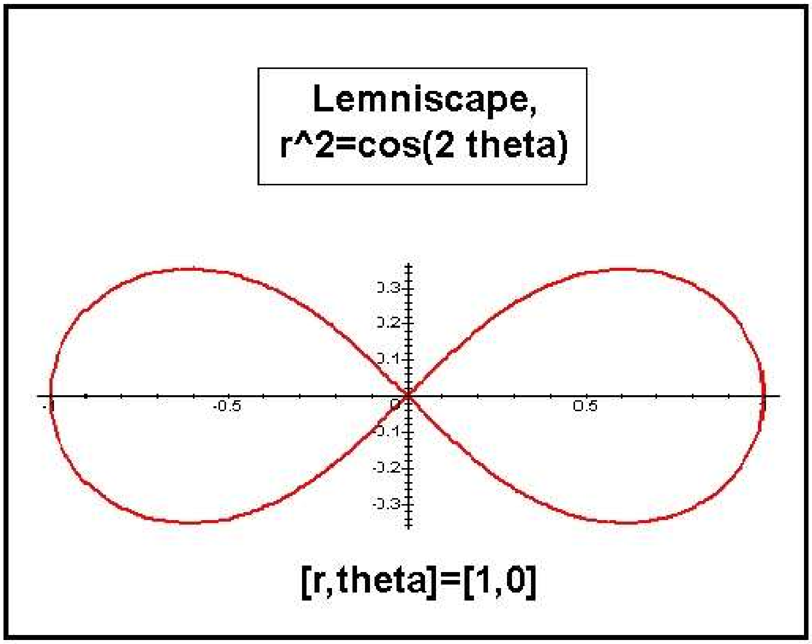
Next consider the first order differential equation-

$$dr/d\theta=-\sin(2\theta)/r \text{ subject to } r(0)=1$$

On separating the variables we have the solution-

$$r^2=\cos(2\theta)$$

Plotting this function we get the bowtie figure shown-

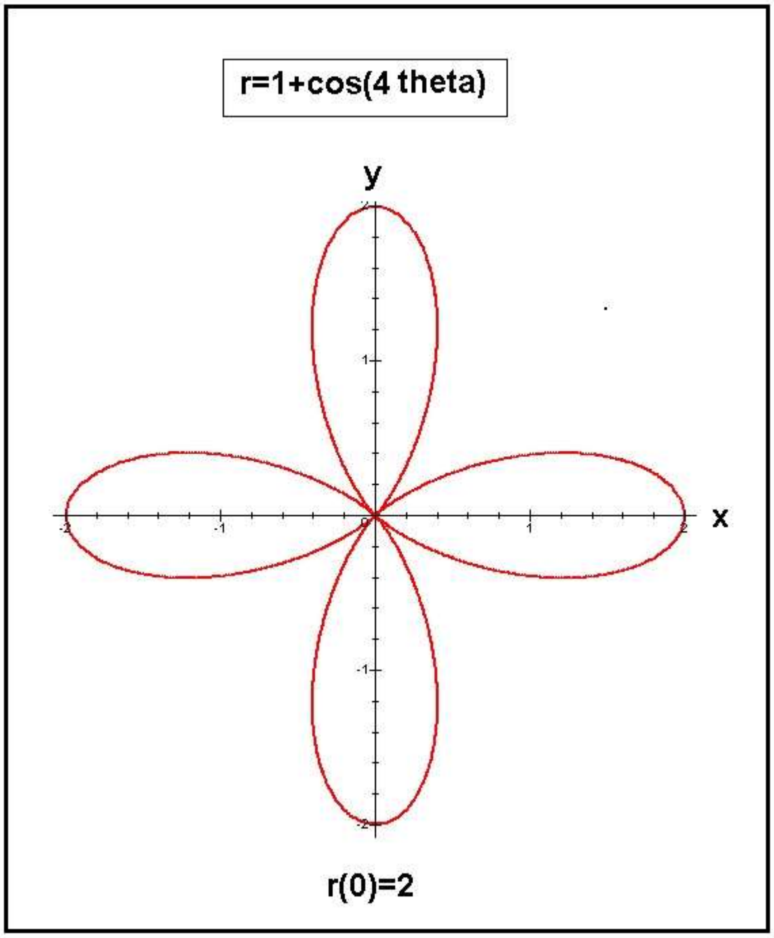


It is referred to in the literature as a Lemniscape. It looks very much like the infinity sign,

The next equation is-

$$dr/d\theta=-4\sin(4\theta) \quad \text{subject to } r(0)=2$$

its solution yields the four petal rose shown-



The total area equals A=8 int[1+cos(4θ),θ=0..π/4]=2π.

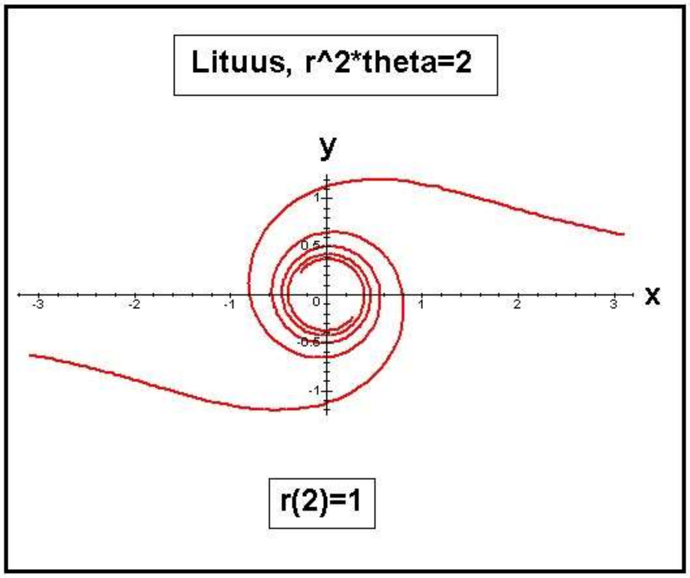
Another 2D curve is governed by the ODE-

$dr/d\theta=-r/2$ subject to r(2)=1

Its solution yields the simple product curve-

$(r^2)\theta=2$

It is referred to as a Lituus and looks as follows-



In Cartesian coordinates the Lituus has the more complicated form-

(x^2+y^2)arctan(y/x)=2

As a final equation representing a 2D curve consider-

dy/dx=[x^2+y^2]/[2y(2-x)]

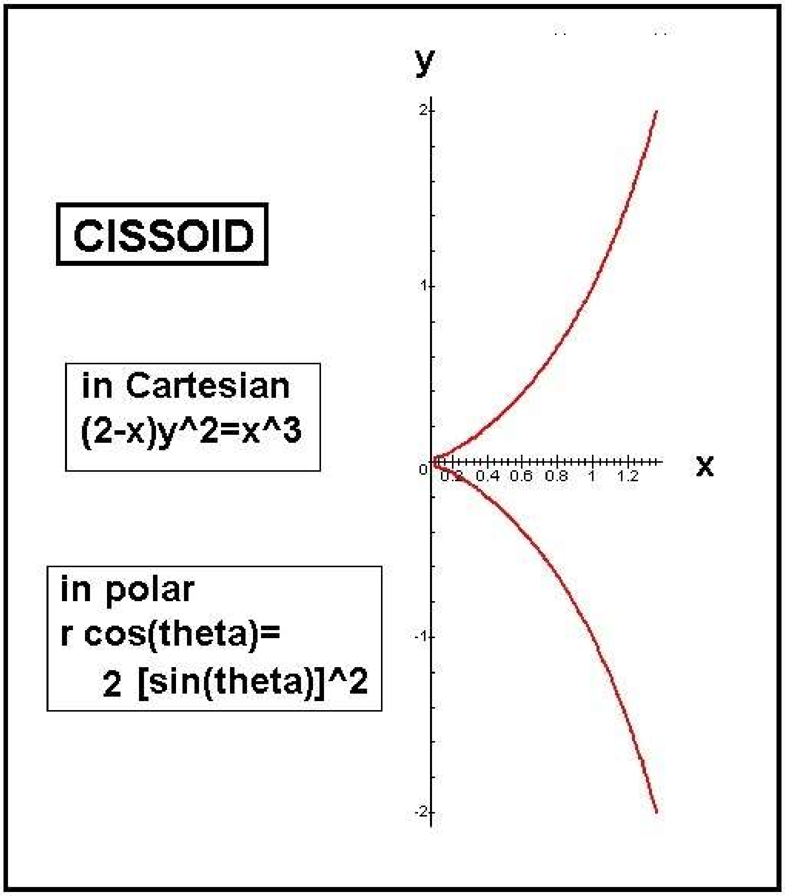
or its equivalent form-

dr/(rdθ)=[4 sin(t)]/r

Of these the second is easier to solve. It yields-

r=2tan(θ)sin(θ) provided we set r(0)=0

Here is its plot-



Although it is not obvious that y becomes unbounded at x=2 from the polar solution, it clearly is indicated by the Cartesian form. The curve has a cusp at x=y=0. It is interesting to note that Isaac Newton played around with the Cissoid and gave its first explicit polar form.

There are an infinite number of other 2D curves which can be generated by first order differential equations but the above give many of the more important ones. The reason this differential equation route is not taken in most high schools and colleges is that analytic geometry precedes calculus in time. So when I look back at my old analytic geometry book no mention of calculus is made.

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