A PRECISE DETERMINATION OF THE GOLDEN RATIO USING THE SOLUTION OF A NON-LINEAR DIOPHANTINE EQUATION

One of the fundamental numbers encountered in mathematics is the Golden Ratio φ defined as-

$$\varphi^2 = \varphi + 1$$

It is an irrational number with the explicit solution-

$$\varphi = [1 + sqrt(5)]/2 = 1.61803398\ldots$$

To get its value to any desired number of digits it is only necessary to know the root of five to about the same number of digits. We wish in this note to derive a rapidly converging infinite series for root five and hence φ to high order of accuracy using the solution of a non-linear Diophantine Equation.

To accomplish this we start with the non-linear Diophantine Equation-

$$y^2=1+5x^2$$

Taking the root and applying a binomial expansion, this equation can be re-written as-

$$\mathrm{sqrt}(5)=(5x/y)\{1+1/(1!\cdot 2^1\cdot 5\cdot x^2)-1/(2!\cdot 2^2\cdot 5^2\cdot x^4)$$
$$+(1\cdot 3)/(3!\cdot 2^3\cdot 5^3\cdot x^6)-(1\cdot 3\cdot 5)/(4!\cdot 2^4\cdot 5^4\cdot x^8)+\ldots\}$$

The integer values of x and y are given by solving the above Diophantine Equation. A simple mathematics program produces the results-

x	Y
4	9
72	161
1292	2889
23184	51841
416020	930249
7465176	16692641
133957148	299537289

As x gets large one notes that y(n)/x(n)=sqrt(5)=2.2360 and y(n+1)/y(n)=17.94427.. Thus, without much effort, a x(8) and y(8) search yields-

x(8)=x=2403763488 and y(8)=y=5374978561

Plugging these last two values of x and y into the above series for sqrt(5) produces, after some manipulations, the result-

$$\text{sqrt(5)}=(5*x)/y)\ \{1+\sum_{k=1}^{\infty}\left(\frac{(2*k-2)!*(-1)^\wedge(k+1)}{k!*(k-1)!*2^{2k-1}*5^{k}*x^{2k}}\right)\ \}$$

From it we have the new rapidly converging series for φ given by-

φ**=(1/2)+(1/2)sqrt(5)**

Taking just the first three terms in the infinite series for sqrt(5) we get-

φ **≈1/2+(5x/2y){1+1/10x^2-1/200x^4}**

This produces the sixty digit accurate result-
φ **≈1.61803398874989484820458683436563811772030917980576286213545**
Here again x is taken as x(8) and y=y(8).The convergence rate is seen to be amazing and will get even more rapid as one uses the Diophantine Solutions x(9) and y(9) and larger.

The present approach works equally well for square roots of different numbers such as sqrt(2) with each integer having its own table of x,y satisfying the Diophantine Equation y^2=1+N x^2.The x,y integers are obtained by using the one line math program-

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for x from 0 to 200 do({x,sqrt(1+N*x^2)})od
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One progressively sets the range on x to encompass the expected integer value for y.

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