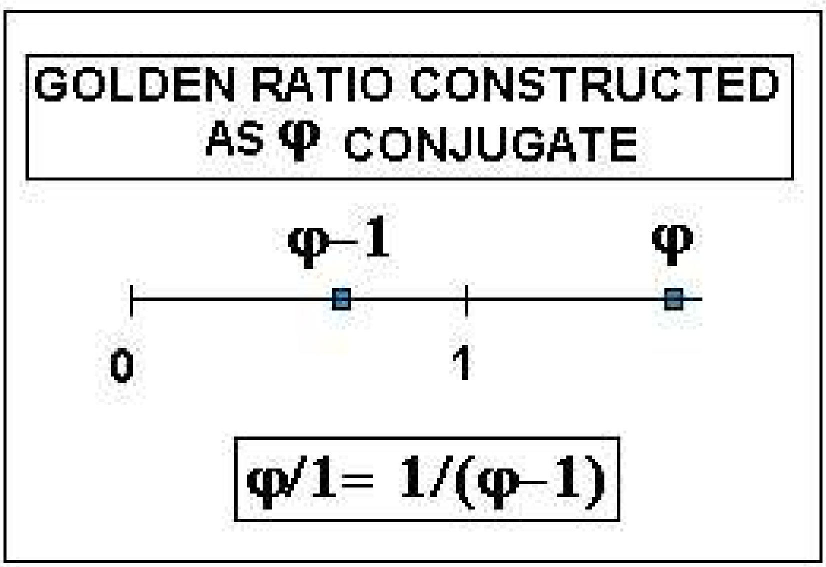
PROPERTIES OF THE GOLDEN RATIO

Several thousand years ago the ancient Greeks discovered a new irrational number-

φ=[1+sqrt(5)]/2=1.6180339…

, now known as the Golden Ratio. Believing that rectangles with this side ratio where the most pleasing to the eye , its dimensions were often incorporated into their temple architecture. Or so it appeared to later archeologists. The number also has many interesting mathematical properties which we wish to discuss in this article.

Let us first derive the equation for the Golden Ratio. We start by looking at the following pattern of two points in blue lying along the x axis.-



The point $(\varphi - 1)$ =[sqrt(5)-1]/2 is the conjugate of $\varphi = [1 + sqrt(5)]/2$. As expected , multiplying $\varphi \; by \; \varphi - 1$ produces unity . Next we take ratios to get-

$\frac{\varphi}{1} = \frac{1}{(\varphi-1)}$ or the equivalent expression $\varphi^2 = \varphi + 1$

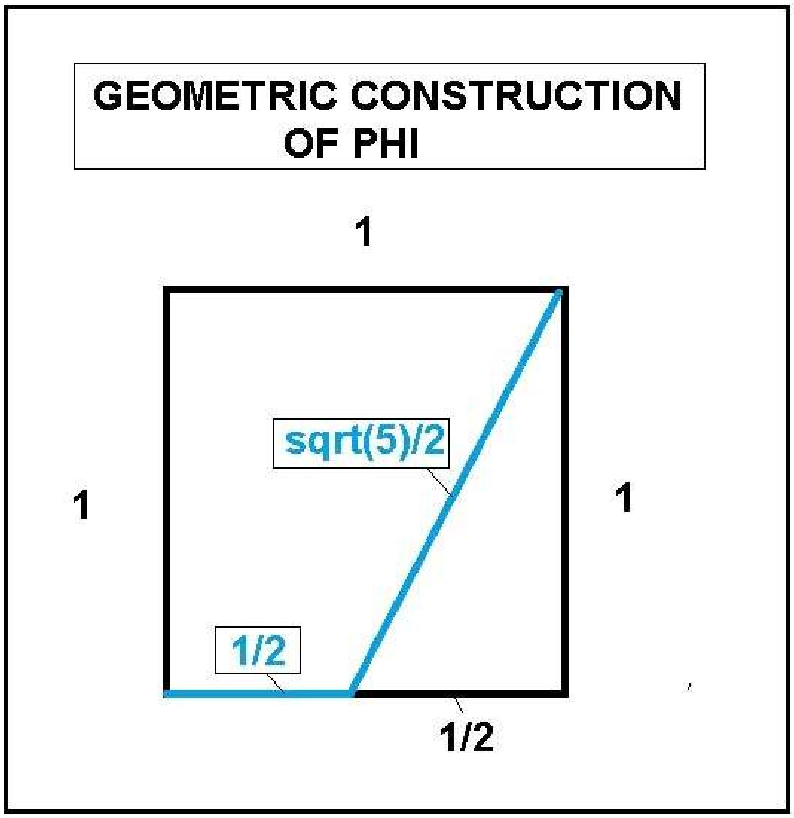
On solving this quadratic we get the positive root-

$$\boldsymbol{\varphi = [1 + sqrt(5)]/2}$$

To ninety-nine place accuracy it reads-

$\boldsymbol{\varphi}$ =1.61803398874989484820458683436563811772030917980576286213544862270526046 281890244970720720418939114…

Geometrically φ can be thought of as the length of the two heavy blue lines in the following square-



A very interesting property of the Golden Ratio is that it can be rewritten as a very simple continued fraction. This fraction is derived as follows. Begin with the basic definition-

$$\varphi = 1 + \frac{1}{\varphi} = 1 + 1/(1+1/\varphi)$$

Continuing on we get the continued fraction-

CONTINUED FRACTION FOR THE GOLDEN RATIO

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 +}}}}$$

This expansion converges rather rapidly. Taking the first three lines, one gets $\varphi \sim \frac{5}{3} = 1.667$.

There are numerous integral solutions involving the Golden Ratio. Whenever the solution to an integral involves the sqrt(5), the integral also can be rewritten as a function of φ. Consider-

$$\int_{x=0}^{1} \frac{dx}{sqrt(4+x)} = \frac{1}{1+\frac{sqrt(5)}{2}} = \frac{2}{1+2\varphi} = 0.4721359 \ldots$$

and-

$$\int_{x=0}^{1/2} 1/sqrt(1+x^2)dx$$ =ln[2/(-1+sqrt(5))]=ln(φ)=0.4812118…

Furthermore we know that the diagonal of a regular pentagon of sides one, yields, by use of the Law of Cosines, that-

$$\varphi = 2\cos(\frac{\pi}{5})=1.6180339\ldots$$

Often in the literature you find questions such as what is the value of φ^n *for any positive integer n*? The answer follows directly from the following table-

n	φ^n
2	1φ +1
3	2φ + 1
4	3φ + 2
5	5φ +3

6	8φ + 5
7	13φ +8
8	$21\varphi + 13$
9	34φ + 21
10	55φ + 34
11	89φ +55
12	144 φ +89

To calculate [(1+sqrt(5))/2]^12 one does not need to carry out a page long derivation. We simply have-

$$\varphi^{12} = 72\big(1 + sqrt(5)\big) + 89 = 161 + 72sqrt(5) = 321.996894 \ldots$$

To extend this table to higher ns we can make use of the equalities-

$$\varphi^{n+1} = (a(n) + a(n-1))\varphi + (b(n) + b(n-1))$$

or-

$$\varphi^{n+1} = (a(n) + b(n))\ \varphi + a(n)$$

So we get a(12)=144, a(11)=89, b(12)=89, and b(11)=55. Hence –

$$\varphi^{13} = (144 + 89)\varphi + (89 + 55) = 233\varphi + 144 = 521.0019\ldots$$

Finally we look a the following infinite series for the Golden Ratio. This reads-

$$\varphi = 1 + \frac{1}{1\cdot1} - \frac{1}{1\cdot2} + \frac{1}{2\cdot3} - \frac{1}{3\cdot5} + \frac{1}{5\cdot8} - \frac{1}{8\cdot13} + \frac{1}{13\cdot21} - \frac{1}{21\cdot34} +$$

Note that all terms in this series are made of ascending Fibonacci Numbers. These read-

f[n]={1,1,2,3,5,8,13,21,…} where f[1]=1,f[2]=1,f[3]=2,f[4]=3

They are constructed by the formula-

f[n+1]=f[n]+f[n-1]

The infinite series representation is given by-

$$\varphi = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(f[n]*f[n+1])}$$

This series converges very slowly and so cannot compete with evaluating φ using the sqrt(5) approach. One also notices in the limit as n goes to infinity, the ratio of f[n+1]/f[n] equals the Golden Ratio φ.

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