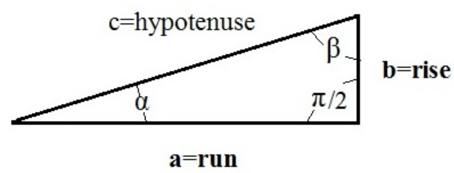
# RELATION BETWEEN GRADE AND ANGLE

It is well known that measurements of the inclination of a surface such as the slant of pyramid surfaces to the design of present day roadways are not expressed directly in terms of angle measured in degrees or radians but rather as a grades(or slope). The following definition sketch shows the relation between angles , incline, and the side length of a right triangle-

DEFINITION SKETCH OF GRADE (=rise/run)



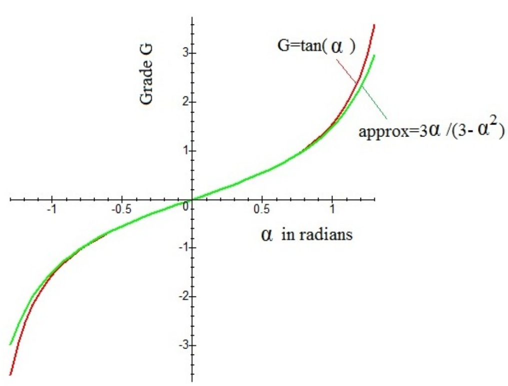

$\alpha$ =arctan(b/a)

The right triangle satisfies the Pythagorean theorem $a^2+b^2=c^2$ . The horizontal sidelength ‘a’is termed the run and the vertical sidelength ‘b’ the rise. The grade of the inclined surface(hypotenuse=c) is given by-

$$Grade = G = \frac{rise}{run} = \frac{b}{a} = \tan(\alpha)$$

Thus a railway bed which rises b=300ft for a run of one mile has an average grade of G= 300/5280=0.05682. In terms of the angle we have $\alpha$ =arctan(0.05682)=0.05676 radians=3.225 degrees. The degree to angle conversion is 180/π=57.2957795 deg/rad. A plot relating grade G to angle $\alpha$ in radians follows-

GRADE VERSUS INCLINE ANGLE



Here we have measured the angle in terms of radians. It is interesting that for small angle we have that-

$$G \approx \frac{3\alpha}{(3-\alpha^2)}$$

where the polynomial quotient on the right is the first approximation to arctan( $\alpha$ ) given by us in an earlier note. For roads such as the Pennsylvania Turnpike, which was built upon an old railway bed, the grade is no more than G=3/100=0.03. The angle of inclination along the entire highway should thus always be less than arctan(G)=0.02999 radians=1.718 degree. Its steepest portion occurs at the Allegheny Mountain Tunnel near Somerset, PA .

Because the Grade is usually quite small for most roads and railbeds , it is convenient to talk in terms of percent grade which equals P=100G. Thus the PA turnpike has a percent grade along its entire length of no more than 3%. Railways throughout the world typically have a P of less than 5% although some in mountainous areas reach as high as 10%. With cog assistance this number can become still higher. Think of the steep inclination associated with the first leg of a typical rollercoaster ride. Here is a table of percent grade P versus inclination angle θ=180 $(\alpha/\pi)$ in degrees-

| Percent Grade, P | Incline Angle in Degrees, θ |
|---|---|
| 1 | 0.5729 |
| 5 | 2.8624 |
| 10 | 5.7106 |
| 20 | 11.3099 |
| 40 | 21.8014 |
| 80 | 38.6598 |
| 100 | 45 |

The formula used in constructing this table is-

$$\theta = \frac{180}{\pi}\arctan(\frac{P}{100})$$

I can also use this formula for defining any other slope such as that of the stairway in my home where each step has a run of 10” and a rise of 7.5”. Thus its incline angle is θ=180 arctan(0.75)=36.87degrees.

Since most grade measurements have relatively small values, one has historically increased their values by certain multiplication factors such as the 100 to define a percent grade. Actually we should call such a modified grade measure the met and define it as-

$$Met = P = \frac{rise\ in\ cm}{run\ in\ m}$$

Here the Met stands for metric as it uses both the centimeter and the meter in its ratio. In a like manner one could introduce a grade measure called the uscu and define it as-

$$Uscu = \frac{rise\ in\ inches}{run\ in\ feet}$$

since it would use US Customary Units. The US is one of the few remaining countries in the world that still uses these archaic units of measure while the rest of the world has long ago adopted the metric SI system of units.

The ancient Egyptians measured their slopes in Sekeds. Using the above triangle, the seked can be defined as-

$$\text{Seked}=\frac{sidelength\ b\ in\ cubits}{sidelength\ a\ in palms}$$

In their system of length measure , 1 cubit=7 palms=28 digits. Thus for a 45-90-45 degree triangle the slope of the hypotenuse would be 7 sekeds which translates to 45 degrees. The Great Pyramid at Giza has slant angle of its four sides set at 38.16 degrees relative to a vertical line passing through its vertex and centerpoint of the base. This corresponds to a value of 5.5 sekeds.