CONSTRUCTION OF THE TRIANGULAR GREEN'S FUNCTION

Consider the BVP y"(x)= -f(x,y) subject to y(0)=y(1)=0 . Integrate both sides of the equation once to get y'(x)=y'(0)+Int[f(t)dt, t=0..x]. Integrating again and using y(0)=0 yields-

$$y(x) = x\ y'(0) - \int_0^x dt \int_0^t f(\zeta) d\zeta$$

Now setting x=1 and reducing the double integral to a single integral, we find y'(0)=x*Int[f( )*(1-)*d which allows one to write-

$$y(x) = x * \int_0^1 (1-\zeta)\ f(\zeta)\ d\zeta \ - \int_0^x (x-\zeta)\ f(\zeta)\ d\zeta$$

Noting that the integration range [0..1] may be broken up into [0..x]+[x..1], one obtains the desired result-

$$y(x) = \int_0^1 G(x,\zeta)\ f(\zeta)\ d\zeta$$

Here G(x,t) is the well known triangular kernel-

$$G^-(x,t) = x(1-t) \quad for\ \ x<t, \quad G^+(x,t) = t(1-x) \quad for\ \ x>t$$

This two part expression for a Green's function will generally be the case when dealing with BVPs. It will turn out that each differential operator and specified boundary conditions will have its own unique Green's function.