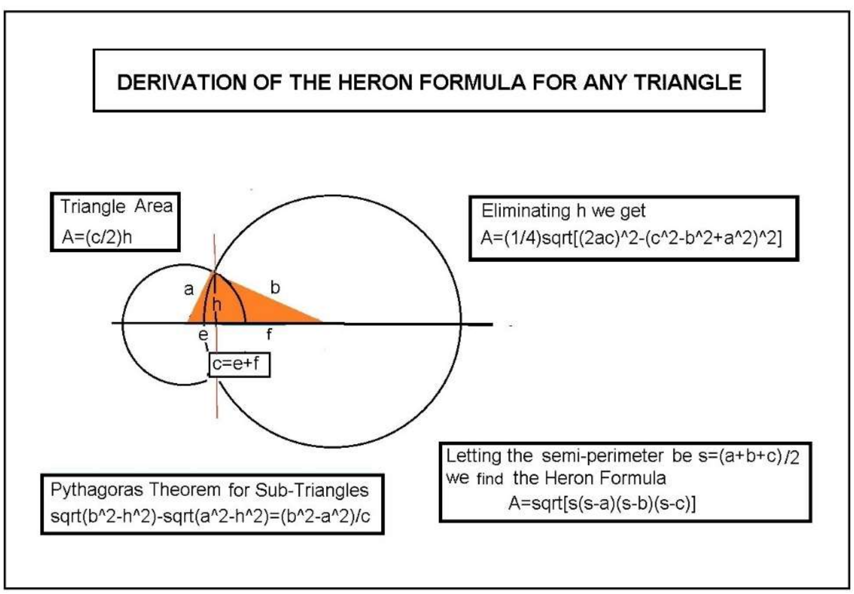
DERIVATIONOF THE HERON FORMULA AND APPLICATION TO N-SIDED POLYGONS

One of the best known, but least often derived formulas for triangle area A , is the Heron Formula

$$A=sqrt[s(s-a)(s-b)(s-c)]$$

, where s=(a+b+c)/2 is the semi-perimeter of any scalene triangle with sides a, b, and c. This formula was first discovered by Heron of Alaxandria(10AD-70AD) several thousand years ago and students are first introduced to it (without proof) during their high school mathematics classes. I remember asking my math teacher at that time of how the formula was derived but she was unable to give me a good answer. The proof is not difficult but does require considerable bookkeeping. We wish in this note to give perhaps the simplest proof for Heron's Formula and then discuss some extentions to the area of n sided irregular polygons as may be encountered during land surveying.

The simplest proof goes as follows. One starts with a horizontal line and draws two circles of different radii at two different points along this line as shown



One next draws a vertical line passing through the two intersections of the circles This produces two sub-right triangles e-h-a and f-h-b of an orange triangle of side-lenths a,b, and c. Using the Pythagorean Theorem, this produces the algebraic expressions-

$$a^2=e^2+h^2 \qquad \text{and} \qquad b^2=f^2+h^2$$

Eliminating h from these equations yields-

$$b^2-a^2=f^2-e^2=c(f-e)$$

This may be re-written as –

$$\sqrt{b^2-h^2}-\sqrt{-a^2+h^2}=(b^2-a^2)/c=\Delta$$

Next solving for h produces the rather lengthy but exact result-

$$h=1/(2\Delta)\sqrt{(2\Delta a)^2-(-\Delta^2+b^2-a^2)^2}$$

$$=\sqrt{a^2-[(c^2-b^2+a^2)/(2c)]^2}$$

Here h is the perpendicular distance from the longest side c. To get the area of the triangle we have-

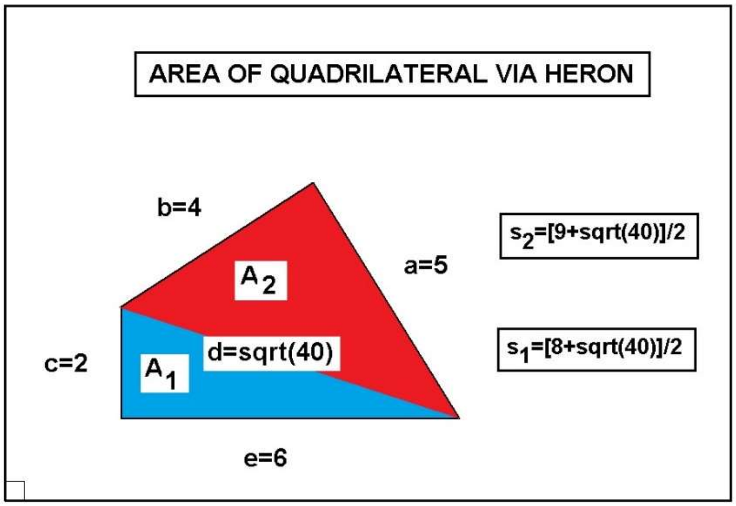
$$A=(c/2)h=(c/2)\sqrt{a^2-[(c^2-b^2+a^2)/(2c)]^2}$$

This is essentially the Heron Formula in disguise. To test the result consider the right triangle a=3, b=4, and c=5. Its area is $A=(5/2)\sqrt{9-[(25-16+9)/10]^2}=6$, according to both the A formula and elementary derivation for the area of any right triangle. We point out in Heron's original derivation his geometrical result gives the answer A=sr, where s is the semi-perimeter $s=(a+b+c)/2$ and r the radius of the largest inscribed circle. For the 3-4-5 triangle it yields r=1. More generally the radius of the largest inscribed circle for any triangle is-

$$r=A/s=\sqrt{[(s-a)(s-b)(s-c)]/s}$$

Working out the numbers for an equilateral triangle of side-length a=b=c=1, we have $s=3/2$, $A=\sqrt{3}/4=0.43301..$, and $r=1/[2\sqrt{3}]=0.28867..$.

Once the area of any triangle has been found using the Heron Formula, it becomes possible to evaluate the area of any larger irregular polygon by dividing the region into n sub-triangles and using Heron's Formula on each of the sub-areas. Let us demonstrate things for the following quadrilateral-



One first divides the area A into two triangular components via a diagonal of length d=sqrt(40). The lower blue portion is a right triangle and has area A_1=6. The red triangle follows from s_2=(a+b+d)/2=[9+sqrt(40)]/2. By Heron's Formula it yields-

$$A_2=(1/4)\text{sqrt}[(81-40)(40-1)]=9.99687\ldots$$

Combining with A_1 we get $A=A_1+A_2$=15.99687… for the full area of the quadrilateral.

We can also extend the area calculations to any n sided irregular polygon by drawing in (n-3) diagonals to produce (n-2) sub-triangles. This fact then allows us to use the Heron Formula (n-2) times to generate the full area. All we need to know is the length of the sides of the polygon to calculate the total internal area.

This fact is of considerable utility for land surveyors to quickly use laser side-length measures to find the area of a parcel of land using the Heron's Formula. Recall that one square mile equals 640 acres. Also one acre equals 43,560 square feet. Consider a parcel of land having the above quadrilateral shape and assume the side-lengths are given in units of 100 ft. This means we multiply 15.99687 by 10^4 to get A=159968.7ft^2=3.67 acres.

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