HYPERBOLIC FUNCTION APPROXIMATIONS

We have shown in several earlier notes on this web page how to use the KTL Method to obtain highly accurate approximations for certain slowly varying functions multiplied by rapidly oscillating Legendre polynomial and integrated over the range 0<x<1. Our most recent discussion of this method is given in article 104 on this Tech-Blog. html page. We want in the present note to look at just one special case involving the KTL Method applied to hyperbolic functions.

We start with the integral-

$$J(n,a)=\int_{x=0}^{1} P(n,x)\cosh(ax)\,dx$$

Here P(n,x) is the nth Legendre Polynomial. These polynomials are given by the Rodrigues's Formula-

$$P(n,x)=[1/(2^n n!)]d^n/dx^n[(x^2-1)^n]$$

They read-

$$P(1,x)= x \;,\; P(2,x)= (3x^2-1)/2 \;,\; P(3,x)=(5x^3-3x)/2 \;,$$

$$P(4,x)=(35x^4-30x^2+3)/8 \;,\; \text{and} \; P(5,x)=(65x^5-70x^3+15x)/8 \; \text{etc}$$

Next we look at a few J(n,a)s starting with n=2 . They read-

$$J(2,a)= (1/a^3)\{(a^2+3) \sinh(a)-(3a) \cosh(a)\}$$

$$J(4,a)=(1/a^5)\{(a^4+45a^2+105) \sinh(a)-(5 a^3+105a) \cosh(a)\}$$

$$J(8,a)-=(1/a^9)\{(a^8+630a^6+51975a^4+945945a^2+2027025) \sinh(a)$$

$$-(36a^7+6930a^5+270270a^3+2027025a) \cosh(a)\}$$

From these we can infer that-

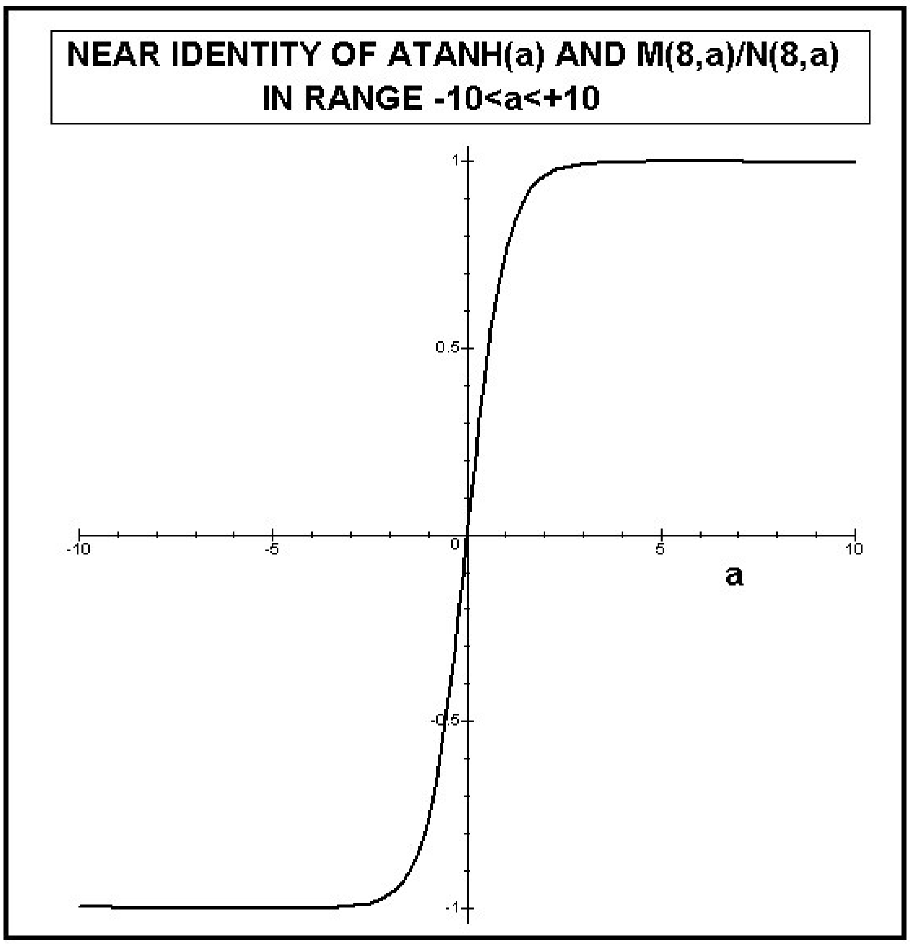
$$J(n,a)=\int_{x=0}^{1} P(n,x)\cosh(ax)\,dx = N(n,a)\sinh(a)-M(n,a)\cosh(a)$$

If we now let a be small and n get large, the last integral approaches zero and we can make the KTL approximation-tanh(a)≈M(n,a)/N(n,a)

For n=8 and a=1, the approximation yields-

tanh(1) ≈2304261/3025576= 0.7615941559557

This agrees with the exact value of tanh(1) to thirteen places. Even better approximations would follow if n is taken to be greater than eight or 'a' less than one. As shown on the following graph, the arctan(a) and the approximation M(8,a)/M(8,a) are nearly identical within the range -10<a<10 -



This arctan curve has found extensive application in the literature for processes

involving the smooth transition from one state to another such as across a shock wave.

Let us now apply the above approximation to any hyperbolic function. We have the basic definitions-

$$\cosh(a)^2=1+\sinh(a)^2$$

From this follows-

$$\sinh(a)=\tanh(a)/\text{sqrt}(1-\tanh(a)^2) \quad \text{and} \quad \cosh(a)=1/\text{sqrt}(1-\tanh(a)^2)$$

If we now let-

$$T=M(8,a)/N(8,a)=(36a^7+6930a^5+270270a^3+2027025a)/$$
$$(a^8+630a^6+51975a^4+945945a^2+2027025)$$

we have a good approximation for tanh(a).

Also we get –

$$\sinh(a)\approx T/\text{sqrt}(1-T^2) \quad , \quad \cosh(a)\approx 1/\text{sqrt}(1-T^2)$$

with-

$$\sinh(2a) \approx 2T/(1-T^2) \quad , \quad \cosh(2a) \approx (1+T^2)/(1-T^2)$$

Evaluating these approximations at a=1 and a=2 produces the following 12 digit accurate results

cosh(1)=1.543080634815

sinh(1)=1,175291193643

cosh(2)=3.762195691083

sinh(2)=3.62686040784

We can also use these results to find an accurate approximation for the base e. One has-

$$\exp(1)=\text{sqrt}\{(1+T)/(1-T)\}\approx 2.7182818284590$$

good to 13 places. To improve over this approximation one needs to involve Legendre Polynomials with more zeros in [0,1]. For the n=8 approximation we have used here there are only 4 zeros of P(8,a) in [0,1].

U.H.Kurzweg
February 12, 2024
Gainesville, Florida