A BRIEF REVIEW OF HYPERBOLIC FUNCTIONS

In our discussion of Laplace Transforms and Complex Variables, we came across many instances involving hyperbolic functions. Since some of you were a bit rusty on the properties of these functions, let me quickly summarize their most important properties. We start with the basic definitions-

$$\sinh(z) = \frac{\exp(z) - \exp(-z)}{2} \quad , \quad \cosh(z) = \frac{\exp(z) + \exp(z)}{2}$$

with z=x+iy. On squaring cosh(z) and subtracting the square of sinh(z), one has the identity-

$$\cosh(z)^2 - \sinh(z)^2 = 1$$

which on dividing yields-

$$1 = \tanh(z)^2 + \sec h(z)^2 = \coth(z)^2 - c\sec h(z)^2$$

Furtheremore, on using the Euler identity exp(z)=exp(x) [cos(y)+isin(y)], we have that-

$$\sinh(iz) = i\sin(z) \quad , \quad \cosh(iz) = \cos(z)$$

Thus if z=π/2, one finds that sinh(iπ/2)=i, sinh(0)=0, cosh(iπ/2)=0, and cosh(0)=1. Differentiating sinh(z) and cosh(z)with respect to z one finds-

$$\frac{d\sinh(z)}{dz} = \cosh(z) \quad , \quad \frac{d\cosh(z)}{dz} = \sinh(z)$$

and integrating, that-

$$\sinh(z) = \int \cosh(z)dz \quad , \quad \cosh(z) = \int \sinh(z)dz$$

Carrying out a standard Taylor series expansion about z=0 one finds –

$$\sinh(z) = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + ... = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)^{2n+1}}$$

and

$$\cosh(z) = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + ... = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$

Thus we have, for example, that the sum of the reciprocal of all factorials yields-

$$e = 2.718281828459045... = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ... = \cosh(1) + \sinh(1)$$

On dividing sinh(z) by cosh(z) one finds the series-

$$\tanh(z) = \frac{\sinh(z)}{\cosh(z)} = z - \frac{z^3}{3} + \frac{2z^5}{15} - \frac{17z^7}{315} + .. = \int_0^z \frac{du}{\cosh(u)^2}$$

provided that $|z|<\pi/2$.

Many integrals can be solved in terms of hyperbolic functions. Consider the integral –

$$K(z) = \int_0^{2z} \frac{dp}{1 + \cosh(p)}$$

Here we let p=2t and then manipulate things to obtain-

$$K(z) = \int_{t=0}^{t=z} \frac{2dt}{1 + \cosh(2t)} = t = 4\int_{t=0}^{t=z} \frac{dt}{\exp(2t) + \exp(-2t) + 2} = \int_0^z \frac{dt}{(\cosh(t)^2} = \tanh(z)$$

We thus have that-

$$\int_0^1 \frac{dx}{[1 + \cosh(x)} = \tanh(1/2) = 0.462117....$$

Differential equations very often have solutions in terms of hyperbolic functions. Thus F""-F=0 is solved by F=A sinh(x)+B sin(x)+C cosh(x)+D cos(x) and G"-4G'+3G=0 has the even solution G=A cosh(3x)+B cosh(x) and the odd solution G=C sinh(3x)+Dsinh(x).

Inverses of hyperbolic functions follow from inverting z=sinh(u) , z=cosh(v), and z=tanh(w) to get arcsinh(z)=u, arccosh(z)=v, and arctanh(z)=w. Next look at the three integrals-

$$F(z) = \int_0^z \frac{dt}{\sqrt{t^2+1}} \quad , \quad G(z) = \int_0^z \frac{dt}{\sqrt{t^2-1}} \quad , \quad H(z) = \int_0^z \frac{dt}{1-t^2}$$

Substituting t=sinh(u), t=cosh(v), and t=tanh(w) into these integrals yields-

$$F(z)=\int_0^{\operatorname{arcsin}h(z)} du=\sinh^{-1} z\,,\; G(z)=\int_0^{\operatorname{arccos}h(z)} du=\cosh^{-1} z \quad , \quad H(z)=\int_0^{\operatorname{arctan}h(z)} du=\tanh^{-1}(z)$$

These identities allow one to use the binomial expansion to rapidly calculate their Taylor series. As an example look at the following expansion for arctanh(z). We have-

$$\tanh^{-1}(z)=\int_0^z \frac{dt}{1-t^2}=\sum_{n=0}^{\infty}\int_0^z t^{2n}dt=\sum_0^{\infty}\frac{z^{2n+1}}{2n+1}=z+\frac{z^3}{3}+\frac{z^5}{5}+\frac{z^7}{7}+...\text{t}$$

provided that |z|<1. From this last result we can conclude that the sum of the reciprocals of all odd integers will equal arctanh(1) and thus be infinite as expected. However when z=1/2 we find-

$$\frac{1}{1(2^{1)}}+\frac{1}{3(2^3)}+\frac{1}{5(2^{5)}}+\frac{1}{7(2^7)}+...=\tanh^{-1}(0.5)=0.549306144...$$

Again one has numerous integrals involving the inverse hyperbolic functions. See if you can verify the following-

$$\int_{t=0}^{t=z}\sinh^{-1}(t)dt=z\ln[z+\sqrt{1+z^2}]-\sqrt{1+z^2}+1$$

Also you might want to show via a canned mathematics program(such as MAPLE) or your hand calculator that-

$$\int_0^1 \frac{dx}{\cosh(x)^6}=0.518342... \qquad , \qquad \int_0^{\infty}\exp[-\cosh(x)]dx=0.42102...$$