ITERATION METODS

One of the simplest ways to determine the roots of integers and the zeros of polynomial equations is the method of iteration. We want in this article to demonstrate the method via several diverse examples. Let us begin with

(1)-ROOTS OF INTEGERS:

Here we start with the mth root of the integer N. We can write this as –

$$N^{(1/m)}=N_o^{(1/m)}\{1+(\varepsilon/N_o)\}^{(1/m)}$$

, where N_0 is the nearest integer to N whose root No^(1/m) is known and $\varepsilon=N-N_o$. Retaining only the first two terms in a Taylor expansion of the term in the curly bracket then yields the iterative formula-

$$f(n)=[(n-1)f(n)^m+N]/[mf(n)^{(m-1)}] \quad \text{subject to } f(0)=c$$

As n goes to infinity the root of N will be given. Let us demonstrate this iteration method for the root of N=2. A simple substitution yields the following, when starting with f(0)=c=1,-

f(1)=3/2=1.5000

f(2)=17/12=1.4166

f(3)=577/408=1.1414215686

The exact value is sqrt2=1.414213562… So the third iteration is already accurate to six decimal places beyond the decimal point. Going on to sqrt(3) starting with f(0)=2, we find-

f(1)=7/4=1.75..

f(2)=97/56=1.73214..

f(3)=18817/10864=1.73205081..

The exact value is sqrt(3)=1.732050808… The third iteration already brings the accuracy to seven places after the decimal point.

A generalization of the square root formula by iteration follows-

$$f(n+1)=\frac{[f(n)^2+N]}{2f(n)}$$

It is solved by the one line computer program-

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f(0}=c;  for n from 0 to 6 do f{n+1):=[f(n)^2+N]/[2*f(n)] od;
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Applying things to the larger integer N=451, we first set f(0)=c=sqrt(441)=21. This yields the iteration results-

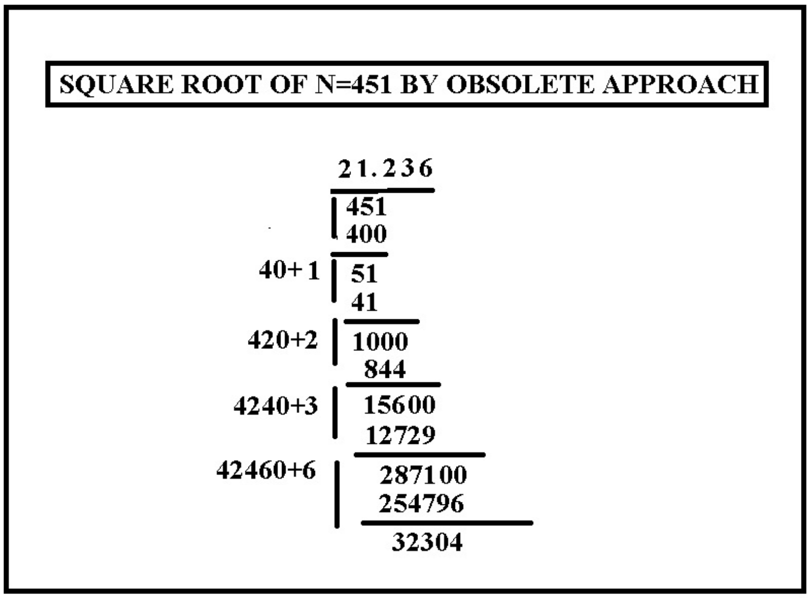
f(2)=446/21=21.23809..

f(2)= 397807/18732=21.236760062..

f(3)=316500817873/14903441448=21.23676058159530134..

The last of these iterates is accurate to sixteen decimal places. The exact value for sqrt(451) is 21.236760581595301302… . What is noticed is that the iterations pick up speed rapidly toward convergence the closer one sets f(0) to sqrt(N).

I point out that there is also an algorithm for finding the square roots different (but related) to the above iteration approach. It was taught to students(like myself) in high school prior to the existence of pocket calculators. We demonstrate its calculation ability for the square root of N=451 –



You will notice that each repeated step yields one extra digit in the value for sqrt(N). So it is quite time consuming when many digits are required. This is not the case with the above iteration approach where the accuracy accelerated rapidly with increasing n.

(2)-ACCURATE VALUES FOR THE GOLDEN RATIO:

One of the more interesting numbers already studied in great detail by the ancient Greeks is the Golden Ratio. Its value φ is derived by looking at a straight line extending from 0 to φ>1. If this line is cut at x=1, we can demand that the following ratios are equal-

$$\frac{\varphi}{1}=\frac{1}{\varphi-1}$$

This is equivalent to a quadratic equation with the closed form solution –

$$\varphi=\frac{1+sqrt(5)}{2}=1.618033989\ldots$$

after some manipulations, an iterative form for the Golden Ratio is found to be-

$$\varphi(n+1) = \frac{[1+\varphi(n)^2]}{2\varphi(n)-1} \quad \text{subject to } \varphi(0) = 1$$

The first few iterates can be done by hand and yield {1,2,5/3,34/21,1597/987}. The convergence to gthe Golden Ratio is quite rapid. Already at the 7th iterate we get the 53 digit long accurate result-

$\varphi(7) =$ **1.61803398874989484820458683436563811772030917980576286**

Many claim that the height to width ratio of many ancient Greek temples satisfy this ratio. My own thought on this proposal is rather skeptical. It is much more likely that the ancient architects built their temples to last and thus emphasized structural integrity. To my own eyes the Golden ratio is not as esthetically pleasing as a ratio of 3 to 2 would be.

(3)-ZEROS OF POLYNOMIALS:

Consider the third order polynomial equation-

y(x)=2x^3-x^2+x-2

It has y(0)=-2 and y(2)=12. So there must be a zero between x=0 and x-2. To find this zero we write down the iteration formula-

f(n+1)=2+f(n)^2-2*f(n)^3 with f(0)=1

Working out the first few terms produces-

f(0)=1

f(1)=1

f(2)=1

Since the iterates f(0) and f(1) are equal to each other we know zero of y(x) occurs at x=f(infinity)=1.

As another polynomial equation consider-

y(x)=x^3-0.5*x^2+x-0.5

To find the zero of this equation we use the iterative formula-

f(n+1)=0.5+0.5*f(n)^2-f(n)^3 subject to f(0)=0 .

It produces-

f(0)=0

f(1)=1/2

f(2)=1/2

So we find the zero of y(x) to occur for x=f(infinity)=1/2. Note we are using here something identical with the Newton-Raphson method. It converges only if one is close enough to the answer with the initial guess f(0). The iteration diverges if one starts the iteration far from the actual zero. Thus if we had taken f(0)=2 things would diverge as f(n)={3,-11/2,182,..} .

We point out that y(x) need not necessarily be a polynomial expression to find its zeros. The present iteration to find zeros will work just as well for any expressions like, for instance,-

$$y(x)=x-\exp(-x)$$

To find the value of x for which y(x) is zero can here be generated by the iteration procedure-

$$f(n+1)=\exp[-f(n)] \quad \text{subject to } f(0)=1$$

The convergence rate is a bit slower than found for polynomial equations. It requires a total of twenty iterations to find x=0.56714 . This is good to only five places of accuracy.

(4)-REVERSING THE PROCEDURE BY GOING FROM f(n+1) TO X:

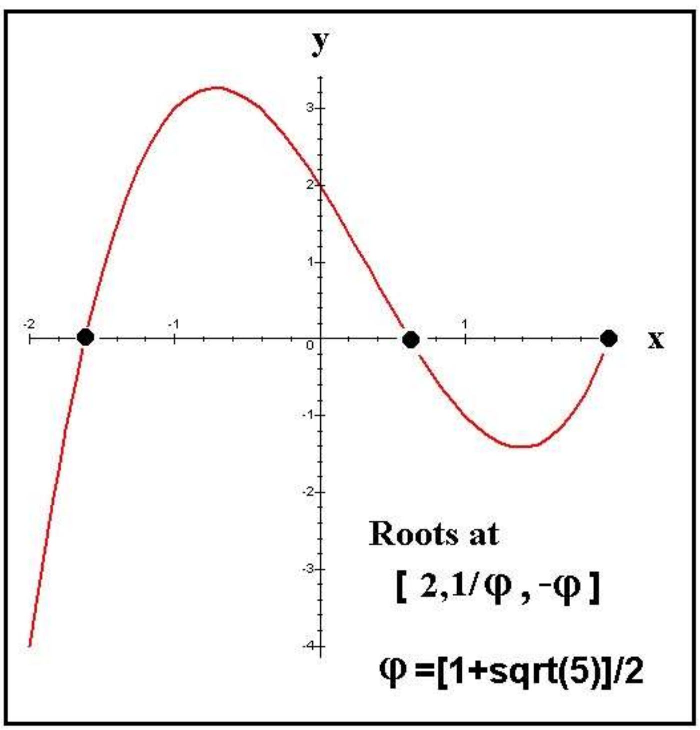
One can also reverse our iteration procedure by starting with f(n+1)=F[f(n)] and reducing this to a function of x whose zeros can be found. Consider the iteration-

$$f(n+1) = [f(n)^3 + 2]/[f(n) + 3]$$

This expression represents the cubic polynomial -

$$y(x)=x^3-x^2-3x+2$$

as shown in the following graph-



The three real roots obtained from this polynomial are x=2, 1/φ, and –φ. Here φ is the Golden Ratio. When applying the iteration to find a zero one must choose an f(0) close to one of the roots . To get the zero at x=1/φ one needs to start the iteration near f(0)=1/2.

We have shown that one can use an iterative procedure to quickly estimate with high accuracy the roots of integers and the zeros of polynomials. To get convergence to the correct answer requires one starts the iteration close to the desired values.

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