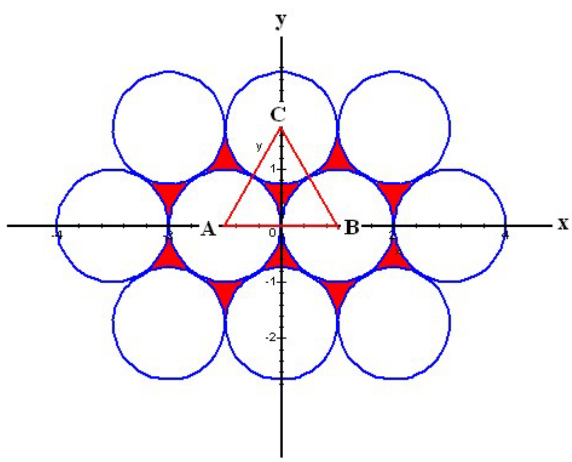
KEPLER'S SPHERE PACKING PROBLEM

Some four hundred years ago(about 1611) Johannes Kepler conjectured that the tightest packing of constant radius spheres occurs for a face centered cubic configuration. That is a configuration where there is an arrangement of parallel rows of spheres such that each sphere just touches its neighboring spheres as shown in the top view of a single layer of this packing -

TOP VIEW OF ONE LAYER OF THE
KEPLER PACKING



A second identical layer is laid on top after being shifted so that its component spheres rest in the depressions shown in red. The height of the second layer center of spheres lies at z=2sqrt(2/3) above the x-y plane as is readily determined by treating the triangle ABC shown as the bottom of a regular tetrahedron. We can draw the full 3D configuration of four of the unit radius spheres centered at

[1, 0, 0], [-1,0,0], [0, sqrt(3), 0], and [0, 1/sqrt(3), 2sqrt(2/3)]

by use of MAPLE commands-

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with(plots);

implicitplot3d( {(x-1)^2 + (y)^2 + (z)^2 = 1, (x+1)^2 + (y)^2 + (z)^2 = 1, (x)^2 + (y-sqrt(3))^2 + (z)^2 = 1, (x)^2 + (y-1/sqrt(3))^2 + (z-2*sqrt(2/3))^2 = 1}, x=-2..2, y=-2..2, z=-2..2.5, grid=[30,30,30]);
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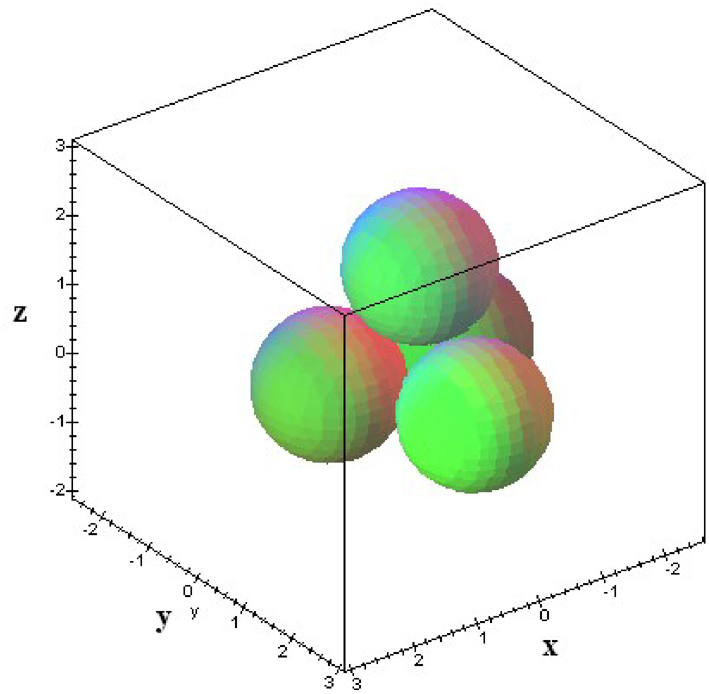
This produces the figure-

FOUR UNIT RADIUS SPHERES STACKED
IN THE KEPLER CONFIGURATION



From this figure and the previous one, one realizes that the effective thickness of each layer is 2sqrt(2/3) and that the number of spheres lying in one plane of length L and width W is nm=WL/[2sqrt(3)]. So we find that the sphere density defined as total sphere volume divided by the volume of the assumed large box LWH is-

$$\eta = \frac{WLH(\frac{4}{3}\pi)}{4\sqrt{\frac{2}{3}}\sqrt{3}\,WLH} = \frac{\pi}{3\sqrt{2}} = 0.740480489693061041169313498345...$$

That is, the voids left between the face-centered packed spheres is about 26%. No other configuration using constant radius spheres has been found where this density is exceeded. It is of course always possible to get higher sphere densities by using different radii spheres with the smaller ones used to partially fill the voids of the present configuration.

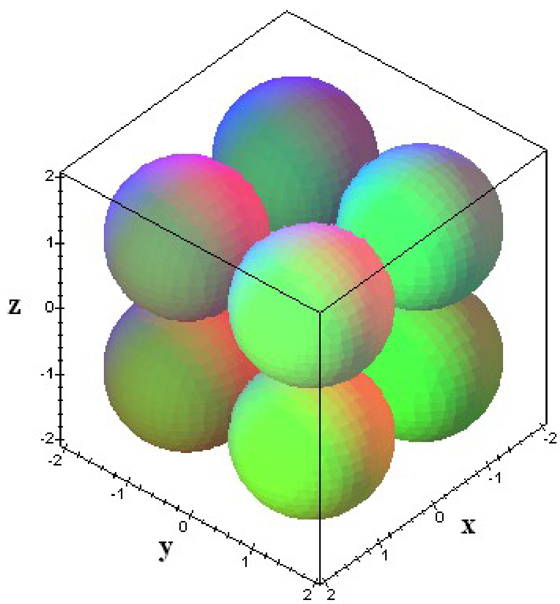
If one were to stack the layers of constant radius spheres directly above each other, the void volume would become larger. In that case the sphere density would be

given as the ratio of sphere volume and the volume of the smallest circumscribed cube. That is-

$$\eta = \frac{\frac{4}{3}\pi R^3}{8R^3} = \frac{\pi}{6} = \mathbf{0.52359877559829887307710723054 8\ldots}$$

A 3D graph for this last configuration of spheres follows-

SPHERES PACKED IN THE $\eta = \pi/6$ CONFIGURATION



We point out the above mathematical results find direct application in the analysis of packed beds in chemical engineering and also have been used in some of our own work on gas removal from bead filled chambers by the application of periodic pressure pulses (see-*"Ventilation of Enclosed Porous Media by the Application of Periodic Pressure Variations",(with M.Jaeger), Int.J.Eng.Sciences 41, 2299-2304, 2003).*

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