USE OF THE KTL METHOD TO FIND HIGHLY ACCURATE APPROXIMATIONS FOR TRIGONOMETRIC FUNCTIONS

About a decade ago while playing with certain definite integrals we observed that-

\[ I(n,a) = \int_{x=0}^{1} \frac{P(2n,x)}{a^2 + x^2} \, dx = \text{Const.} \left\{ M(n,a) - (\arctan(a)/a)N(n,a) \right\} \]

where \( M \) and \( N \) are polynomials in \( a \) at fixed \( n \) and \( P(2n,x) \) the even Legendre polynomial. Upon setting \( a=1 \), we noted that-

\[
\begin{align*}
I(1,1) &= \frac{3}{2} - \frac{\pi}{2} = -0.0707963 \\
I(2,1) &= \frac{20}{3} + \frac{17\pi}{8} = 0.0092177 \\
I(3,1) &= \frac{161}{5} - \frac{41\pi}{4} = -0.0013246 \\
I(4,1) &= -\frac{5728}{35} + \frac{1667\pi}{32} = 0.0001994
\end{align*}
\]

That is, the integral \( I(n,1) \) becomes smaller and smaller as \( n \) is increased, leaving us with the limit-

\[
\lim_{n \to \infty} I(n,1) = 0
\]

So even when \( n \) is not infinite, we can set the integral \( I(n,1) \) for finite \( n \) to zero to get an approximation for \( \pi = 4 \arctan(1) \). At \( n=10 \) this yields the 14 place accurate result-

\[ \pi \approx 3.14159265358979 \]

The reason this approximation works as well as it does with a minimum effort is that we are dealing with a definite integral whose integrand consists of the product of a slowly varying function \( g(a,x) \) with a rapidly oscillating Legendre Polynomial with \( n \) zeros in the range \( 0<x<1 \). This means that any integral of this product is approximately zero. Care must be taken in choosing the even or odd symmetric forms of the Legendre Polynomial. Thus \( g(a,x)=1/(a^2+x^2) \) must be multiplied by \( P(2n,x) \) while \( g(a,x)=\sin(ax) \) must be multiplied by \( P(2n+1,x) \).

The generalized form for the above procedure is now referred to in the literature as the KTL method. ( see [https://wiki.tcllang.org/page/Trig+Procedures+for+degree+measures+as+sind%2C+cosd%2C+tand%2Cetc](https://wiki.tcllang.org/page/Trig+Procedures+for+degree+measures+as+sind%2C+cosd%2C+tand%2Cetc) ) and ( [https://mae.ufl.edu/~uhk/KTL-METHOD.pdf](https://mae.ufl.edu/~uhk/KTL-METHOD.pdf) ). The letters stand for Kurzweg, Timmins and Legendre. The technique can be summarized as
\[ I(n,a) = \int_{x=0}^{1} P(n,x)g(a,x) = \text{Const} \{ M(n,a) - N(n,a)h(a) \} \]

where \( I(n,a) \) is set to zero when \( n >> 1 \), the correct symmetry for \( P(n,x) \) is chosen, and \( h(a) = \int_{x=0}^{1} g(a,x)dx \).

The result is an approximation for \( h(a) \) which improves with increasing \( n \). We point out that the technique could use other rapidly oscillating functions, such as the Chebyshev Polynomials, but none of these have been found to be as effective as those of Legendre.

It is the purpose of this article is to apply the KTL method to obtain highly accurate approximations for all trigonometric functions in the range \( 0 < a < \pi/4 \). Once these approximations are known over this limited range, simple trigonometric identities can then be applied for any point outside this range. Part of the results to be presented below were already known to us back in 2011 and discussed on this web page under the title- POLYNOMIAL QUOTIENT APPROXIMATIONS FOR TRIGONOMETRIC FUNCTIONS.

Just two integrals will be used in the KTL method to obtain approximations for all the trigonometric approximations good to at least 12 digit accuracy. These integrals are:

\[ I(n,a) = \int_{x=0}^{1} P(2n,x) \cos(ax)dx \quad \text{and} \quad J(n,a) = \int_{x=0}^{1} P(2n+1,x) \sin(ax)dx \]

Note the required even symmetry of the integrands. The method would not work if the integrand were odd. To get approximations for \( \sin(a) \), \( \cos(a) \), \( \csc(a) \) and \( \sec(a) \) we start with the found approximation for \( \tan(a) \) taken to a chosen digit accuracy and then make use of the identities:

\[ \cos(a) = \frac{1}{\sec(a)} = \frac{1}{\sqrt{1 + \tan(a)^2}} \quad \text{and} \quad \sin(a) = \frac{1}{\csc(a)} = \frac{\tan(a)}{\sqrt{1 + \tan(a)^2}} \]

The evaluation of the \( I(n,a) \) and \( J(n,a) \) integrals are a simple matter. To get the tan approximation \( T(n) = \sin(a)/\cos(a) \) for a given \( n \) also is straightforward. The values for \( T(n) \) are found to be-
\[ T(1) = a \]
\[ T(2) = \frac{3a}{3 - a^2} \]
\[ T(3) = \frac{a(15 - a^2)}{15 - 6a^2} \]
\[ T(4) = \frac{a(105 - 10a^2)}{105 - 45a^2 + a^4} \]
\[ T(5) = \frac{a(945 - 105a^2 + a^4)}{945 - 420a^2 + 15a^4} \]
\[ T(6) = \frac{a(10395 - 1260a^2 + 21a^4)}{10395 - 4725a^2 + 210a^4 - a^6} \]
\[ T(7) = \frac{a(135135 - 17325a^2 + 378a^4 - a^6)}{135135 - 62370a^2 + 3150a^4 - 28a^6} \]
\[ T(8) = \frac{a(2027025 - 270270a^2 + 6930a^4 - 36a^6)}{2027025 - 945945a^2 + 51975a^4 - 630a^6 + a^8} \]
\[ T(9) = \frac{a(34459425 - 4729725a^2 + 135135a^4 - 990a^6 + a^8)}{34459425 - 16216200a^2 + 945945a^4 - 13860a^6 + 45a^8} \]
\[ T(10) = \frac{a(654729075 - 91891800a^2 + 2837835a^4 - 25740a^6 + 55a^8)}{654729075 - 310134825a^2 + 18918900a^4 - 315315a^6 + 1485a^8 - a^{10}} \]

The accuracy of these T(n) approximations for \( \tan(a) \) increase with increasing n and decreasing ‘a’. As ‘a’ goes toward zero each of the above T(n)s approaches \( T(n) \rightarrow a \). On the other hand at \( a=1 \), which corresponds to one radian, we find the value of \( T(n) \) to be:

<table>
<thead>
<tr>
<th>n</th>
<th>T(n) at a=1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3/2=1.5</td>
</tr>
<tr>
<td>3</td>
<td>14/9=1.55</td>
</tr>
<tr>
<td>4</td>
<td>95/61=1.557</td>
</tr>
<tr>
<td>5</td>
<td>841/540=1.557407</td>
</tr>
<tr>
<td>6</td>
<td>9156/5879=1.55740772</td>
</tr>
<tr>
<td>7</td>
<td>118187/75887=1.5574077246</td>
</tr>
<tr>
<td>8</td>
<td>1763649/1132426=1.55740772465</td>
</tr>
<tr>
<td>9</td>
<td>29863846/19175355=1.557407724654902</td>
</tr>
<tr>
<td>10</td>
<td>565649425/363199319=1.557407724654902230</td>
</tr>
</tbody>
</table>

In this table we have cutoff all those digits where the tan approximation departs from the exact value of \( \tan(1)=1.55740772465490223050697480746… \) We see that \( T(10) \) at \( a=1 \) is accurate to eighteen places.
while T(6) for a=1 is accurate to only eight places. To get values corresponding to any angle θ one needs to use the identity \( a = (\pi/180) \theta \) so that a=1 means \( \theta = 57.2957795\text{ deg.} \).

Having obtained the approximations for T(n) for n=1 through 10 we can now obtain an excellent approximation for any of the trigonometric function at any point in \( 0 < a < \pi/4 \). Values outside this range are then readily found using trig identities as will be shown below.

Let us begin by comparing \( \tan(\pi/6) \) with the intermediate approximation T(6). We find –

\[
\tan(\pi/6) = 1/\sqrt{3} = 0.577350269189625764509148780503 \quad \text{compared to} \quad T(6) = 0.577350269189
\]

The T(6) approximation is accurate to 12 places at \( a = \pi/6 \) while to only 8 places at a=1. This again confirms that the T(n) approximation yields progressively more accurate results as ‘a’ gets smaller.

Look next at the approximate value for sec(a) = 1/cos(a) over the range -7 < a < 7 using T(6).

The curve agrees visually with sec(a) everywhere in \( |a| < 5 \).

We next evaluate all the trigonometric functions at \( a = \pi/8 \) (22.5 deg) again using the intermediate approximation value of T(6) which still has a relatively simple form easily manipulated. The results are terminated when the approximations first depart from their exact values -

<table>
<thead>
<tr>
<th>FUNCTION</th>
<th>APPROXIMATION VALUE AT a=(\pi/8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tan(a)</td>
<td>0.41421356237309 (14 places)</td>
</tr>
<tr>
<td>Cot(a)</td>
<td>2.414213562373 (12 places)</td>
</tr>
<tr>
<td>Cos(a)</td>
<td>0.92387953251128 (14 places)</td>
</tr>
<tr>
<td>Sec(a)</td>
<td>1.08239220029239 (14 places)</td>
</tr>
<tr>
<td>Sin(a)</td>
<td>0.38268343236508 (14 places)</td>
</tr>
<tr>
<td>Csc(a)</td>
<td>2.6131259297527 (13 places)</td>
</tr>
</tbody>
</table>
To find the values of these trigonometric functions for other \( a \)'s inside the range \( 0 < a < \pi/4 \) just involves a reevaluation of \( T(n) \). For outside the range, where the KTL approach becomes less accurate, one makes use of the tangent identity:

\[
\tan(A - B) = \frac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}
\]

On letting \( A = (\pi/4 - \Delta) \) and \( B = (\pi/4 + \Delta) \), this identity becomes:

\[
\tan(A) = \frac{1 - \tan(\Delta)}{1 + \tan(\Delta)} \quad \text{and} \quad \tan(B) = \frac{1 + \tan(\Delta)}{1 - \tan(\Delta)}
\]

This implies that \( \tan(A) = 1/\tan(B) \). So if \( A \) is smaller than \( a = \pi/4 \) rad, where the \( \tan(A) \) approximation has high accuracy, one can use the above formulas to get equal accurate approximations for \( \tan(B) \). Let us demonstrate this for \( \tan(A) = \tan(\pi/8) = T(6) \). Recalling that for this case \( T(6) = 0.41421356237309 \), we find:

\[
\tan(B) = \tan(3\pi/8) = \frac{1}{\tan(\pi/8)} \approx \frac{1}{T(6)} = 2.41421356237309
\]

Thus if we have found a good approximation for \( \tan(A) \) when \( a < \pi/4 \) by the KTL method we have an equally accurate result for \( \tan(B) \) where \( a > \pi/4 \). This applies not only for \( \tan(a) \) but equally well for \( \sin(a) \), \( \cos(a) \), etc. So, for example, when \( A = \pi/6 = T(6) \), we find \( B = \pi/3 \) satisfies \( \tan(B) = 1/T(6) \). Also \( \cos(A) = 1/\sqrt{1 + T(n)^2} \) becomes \( \cos(B) = T(n)/\sqrt{1 + T(n)^2} \). Here is a short 18 place table created by the present approach using the \( T(10) \) approximation:

<table>
<thead>
<tr>
<th>( a ) (rads)</th>
<th>( \tan(a) \approx T(10) )</th>
<th>( \cos(a) \approx 1/\sqrt{1 + T(10)^2} )</th>
<th>( \sin(a) \approx T(10)/\sqrt{1 + T(10)^2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi/6 )</td>
<td>0.577350269189625764</td>
<td>0.866025403784438646</td>
<td>0.500000000000000000</td>
</tr>
<tr>
<td>( \pi/4 )</td>
<td>1.000000000000000000</td>
<td>0.707106781186547524</td>
<td>0.707106781186547524</td>
</tr>
<tr>
<td>( \pi/3 )</td>
<td>1.732050807568877293</td>
<td>0.500000000000000000</td>
<td>0.866025403784438646</td>
</tr>
</tbody>
</table>

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