LAGRANGE MULTIPLIERS

In our above variational methods course we briefly discussed Lagrange Multipliers and showed how these may be used to find the extremum of a function $F$ subject to a set of constraints. We want to here discuss this procedure in more detail and work out several more specific examples of possible interest to the readers. Consider a function of $n$ variables given as-

$$F(x_1,x_2,x_3,\ldots x_n) \text{ plus constraints } g_1(x_1,x_2,x_3,\ldots x_n)=c_1, \ g_2((x_1,x_2,x_3,\ldots x_n)=c_2, \ldots \text{etc}$$

where the $c_n$s are constants. Geometrically one can think of $g_n=c_n$ as hypersurfaces which intersect in a common curve $C$. The gradients of the various $g$ surfaces will be at right angles to the intersection curve as shown-

Here $g_1=g$ and $g_2=h$ in order to simplify the discussion. Note next that the directional derivative of the function $F$ to be extremized (and hence have $dF/ds=0$) is-

$$\frac{dF}{ds} = \nabla F \cdot \tau \text{ where } \tau \text{ is the unit length tangent vector along curve } C$$

Thus we have that the gradient of $F$ is also perpendicular to curve $C$ and hence that $\nabla(g), \nabla(h)$ and $\nabla(F)$ are coplanar. This means in general that-

$$\nabla[F + \lambda_1 g_1 + \lambda_2 g_2 + \lambda_3 g_3 + \ldots] = 0$$

where the $\lambda$s are the Lagrange Multipliers. These $n$ equations plus the equations for the constraints constitute sufficient information to find all $\lambda$s plus determine the extremum value for $F$. 
Let's consider a few examples starting with the simple problem of determining the maximum volume $V$ contained in a cylinder of radius $R$ and height $H$ for fixed surface area $S$. Here one has:

$$
\nabla \left[ \pi R^2 H + \lambda_1 (2\pi R^2 + 2\pi R H) \right] = 0
$$

Thus one finds the following three equations:

$$
RH + \lambda_1 (2R + H) = 0, \quad R + 2\lambda_1 = 0, \quad S = 2\pi R (R + H)
$$

with solutions:

$$
\lambda_1 = -R/2, \quad H = 2R, \quad S = 6\pi R^2
$$

The result states that for maximum possible storage a can should have its diameter just equal to its height. There is an interesting experiment carried our by child psychologists in which they fill both a tall and a wide drinking glass full of the same volume of fruit juice and then ask a child which glass contains the larger amount of fluid. Invariably the child will choose the taller glass. Directly related to this maximum volume problem of a cylinder is the Archimedes observation that the maximum volume sphere which can be put in a cylinder requires that the cylinder height just equals its diameter. Under those conditions one has that the sphere to cylinder volume is exactly $2/3$. A graph of this geometry (and one related to that supposedly engraved on Archimedes’s tombstone) is:

![Sphere in a Cylinder Diagram](image)

Next let us ask what is the radius of the largest volume rectangular solid which can fit into a unit radius ($R=1$) sphere. Here the Lagrange Multiplier method produces:
\[ \nabla [xyz + \lambda_1 (x^2 + y^2 + z^2)] = 0 \]

or the equivalent-

\[ yz + 2\lambda_1 (x) = 0 , \quad xz + 2\lambda_1 (y) = 0 , \quad \text{and} \quad xy + 2\lambda_1 (z) = 0 \]

These yield the solutions \( x=y=z=1/\sqrt{3} \). That is, the largest volume rectangular solid capable of fitting into a unit radius sphere is a cube with sides of length \( 2/\sqrt{3} \). We show you here a 3D picture of this result constructed via MAPLE-

**CUBE INSIDE A SPHERE**

Note that the ratio of cube volume \( V_c=8x^3 \) to that of the sphere volume \( V_s=4\pi/3 \) is 
\[ 2/[\pi \sqrt{3}] = 0.3675.. . \]

As a third, more difficult, example, consider the shortest distance from the parabola \( y=1+x^2 \) and the circle \((x-2)^2+y^2=1=0\). This time one wants to extremize the distance squared between the two curves subjected to the curve constraints. One has the following picture-
We find-

\[ \nabla [ (x_1 - x_2)^2 + (y_1 - y_2)^2 + \lambda_1 (1 + x_1^2 - y_1) + \lambda_2 ((x_2 - 2)^2 + y_2^2 - 1)] = 0 \]

which produces the four equations-

\[ 2(x_1 - x_2) + 2x_1 \lambda_1 = 0, \quad -2(x_1 - x_2) + 2\lambda_2 (x_2 - 2) = 0, \quad 2(y_1 - y_2) - \lambda_1 = 0, \]
\[ \text{and} \quad -2(y_1 - y_2) + 2\lambda_2 y_2 = 0 \]

These, when used in conjunction with the constraints \( y_1 = 1 + x_1^2 \) and \( (x_2 - 2)^2 + y_2^2 = 1 \), yields, after elimination of the \( \lambda_1 \)s and \( y_1 \) and \( y_2 \), the rather nasty set of two highly non-linear algebraic equations-

\[ 2x_1[(1 + x_1^2) - \sqrt{1 - (x_2 - 2)^2}] + (x_1 - x_2) = 0 \quad \text{and} \]
\[ (x_1 - x_2)\sqrt{1 - (x_2 - 2)^2} - (x_2 - x_1)[(1 + x_1^2) - \sqrt{1 - (x_2 - 2)^2}] = 0 \]

They can be solved graphically and one finds –

\[ x_1 = 0.5536.., \quad x_2 = 1.2579.., \quad y_1 = 1.30640.., \quad \text{and} \quad y_2 = 0.67029.. \]

The results also show that the minimum distance between the curves will be-

\[ \delta = 0.94905.. \]
Note from the above figure that this shortest route between the constraint curves occurs where their slopes are equal, a fact making possible a rapid evaluation of the above equations since the figure allows one to make good initial estimates for the expected values.

As a final example of a Lagrange Multiplier application consider the problem of finding the particular triangle of sides a, b, and c whose area is maximum when its perimeter L=a+b+c is fixed. Our starting point here is Heron’s famous formula for the area of a triangle:

\[ A = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{where} \quad s = (a + b + c)/2 = L/2 \quad \text{as the half perimeter} \]

A little manipulation allows one to recast this result in the form:

\[ A^2 = \frac{[(a + b)^2 - c^2][c^2 - (a - b)^2]}{16} \]

Next applying the Lagrange Multiplier method we have:

\[ \nabla \left[ (x + y)^2 - z^2 \right] \left[ z^2 - (x - y)^2 \right] + \lambda [x + y + z] = 0 \]

or the equivalent three algebraic expressions:

\[ \begin{align*}
2(x + y)[z^2 - (x - y)^2] - [(x + y)^2 - z^2]2(x - y) + \lambda &= 0, \\
2(x + y)[z^2 - (x - y)^2] + [(x + y)^2 - z^2]2(x - y) + \lambda &= 0, \\
-2z[z^2 - (x - y)^2] + [(x + y)^2 - z^2]2z + \lambda &= 0
\end{align*} \]

Eliminating \( \lambda \) one has \( (x+y)U-V(x-y)=(x+y)U+V(x-y)=zU+Vz \) with \( U=z^2-(x-y)^2 \) and \( V=(x+y)^2-z^2 \). Combining the first two we find \( 2(x-y)V=0 \) so that \( x=y \). Combining the second and the last one has \( z(V-U)=2xU \) or \( (z+2x)(z-x)=0 \) implying that \( z=x=y \). Thus one may conclude that the triangle of largest area subjected to the constraint of a fixed perimeter is an equilateral triangle. Using the Heron area formula, one has:

\[ A = \frac{1}{4} \sqrt{3a^2} \left[ a^2 \right] = \frac{a^2 \sqrt{3}}{4} \]

for the largest possible area triangle for fixed perimeter \( L=3a \).